TIGHT BINDING BOOK

DIFFERENTIAL AND INTEGRAL CALCULUS

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PREFACE TO FIRST EDITION

This text on the Differential and Integral Calculus is presented with the belief that it is well adapted for use both in academic colleges and in engineering schools.

The student who studies this subject because of his attraction to mathematics is not well equipped if he lacks a fair appreciation of the wide applications of the Calculus in modern science and in engineering. On the other hand, the student who is required to use the Calculus in some chosen field of science can make more intelligent and extensive applications if he understands the underlying principles of the subject. Hence, whether mathematics is to be regarded as the queen of the sciences or as the tool of the scientist, the study of the Calculus for the future teacher of mathematics and for the future engineer should differ only in the degree of emphasis placed on the theory and the applications.

As is well known, a complete rigorous proof of some of the theorems in the Calculus is out of the question for the beginning student, whereas the applications are easily made and are of extreme importance. With this in mind, the authors have endeavored to use only proofs which are valid but which may involve certain assumptions, the proof of which belongs properly in an advanced course. These assumptions are pointed out to the student; for at this stage he should be instructed to examine proofs more critically, in order that he may realize some of the difficulties to be encountered, and also that he may avoid the common pitfalls.

A comprehensive review of as much analytic geometry as is required in the Calculus has been included in Chapters I, II, and VIII. These may be omitted, or they may be used only for reference, at the discretion of the teacher.

The authors are sincerely grateful to Professor W. A. Wilson, who has kindly made many pertinent suggestions; to Professor

O. T. Geckeler, who has contributed many problems and valuable suggestions, and who has prepared the material for several important parts of the text; and to the Macmillan Company, the publishers, who have been most considerate in cooperating with the authors and the editor in the publication of this text.

J. H. NEELLEY
J. I. TRACEY

August, 1932

PREFACE TO SECOND EDITION

In revising this book, special attention has been given to the selection and arrangement of problems. To increase classroom utility many additional carefully chosen problems have been interspersed throughout the text; especially those of a less involved nature, for purposes of drill, and on the other hand, a number of highly challenging problems.

Although the general plan and organization of the book remains unchanged, new material has been added in Chapters VII, VIII, and IX and some other sections have been rewritten.

The authors are grateful for helpful suggestions from the many teachers familiar with the earlier edition, and to those who have manifested an interest in the revision by constructive criticism.

J. H. N.
J. I. T.

June, 1939

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DIFFERENTIAL AND INTEGRAL CALCULUS

CHAPTER I

COORDINATE SYSTEMS — GRAPHS

1. Coordinate Systems. A system of coordinates is a method of representing the position of points by means of numbers. In the elementary geometry of the plane two systems are in general use, ***rectangular coordinates*** and ***polar coordinates***.

2. Rectangular Coordinates. To establish a system of rectangular coordinates in a plane, it is necessary to draw a pair of perpendicular lines, called axes, and to have an appropriate unit of length.*

The perpendicular lines are the ***x axis*** and the ***y axis*** and their point of intersection is the ***origin***. It is customary to draw the x axis horizontally and the y axis vertically; the four quadrants into which they divide the plane are numbered as in trigonometry.

The position of any point in the plane is designated by two real numbers, called ***coordinates***, which represent *the respective distances from the axes to the point* as measured in terms of the given unit. These coordinates are called ***abscissa*** and ***ordinate***.

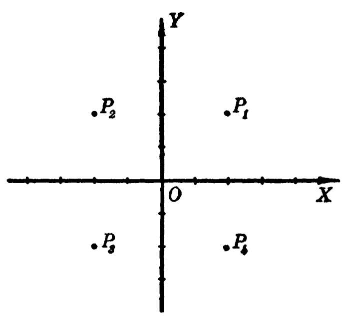


FIG. 1

The ***abscissa*** of a point is *the distance from the y axis to the point.*

The ***ordinate*** of a point is *the distance from the x axis to the point.*

If the coordinates of a point are x and y, respectively, they are written in the form (x, y), the first number always being the abscissa.

In any system of coordinates it is essential that a given pair of numbers shall designate one and only one point. Hence it is necessary to distinguish between the coordinates of such points as

* There are coordinate systems in which the axes are not perpendicular, but little use is made of oblique axes in elementary geometry.

represented in the figure; this is done by making abscissas and ordinates directed line-segments.

If AB is a directed line-segment, then BA is $-AB$, that is, reading a segment in the opposite direction changes its sign. Thus, if the length of the segment BA is considered as $+6$ units, then AB is -6 units. The signs for directed segments in rectangular coordinates are usually as follows:

A horizontal segment when read from left to right is positive, if read from right to left it is negative.

A vertical segment when read upward is positive, if read downward it is negative.

If an oblique segment is directed, it is positive when read upward and negative when read downward.

However, we shall use an arrow head to designate the positive direction along each axis, as it is sometimes more convenient to reverse the positive direction along one of the coordinate axes.

To ***plot a point***, when its rectangular coordinates are given, is to mark its position in the plane with reference to the coordinate axes. Thus the point $(4, -3)$ is the point 4 units from the y axis in the positive direction of the x axis and 3 units from the x axis in the negative direction of the y axis.

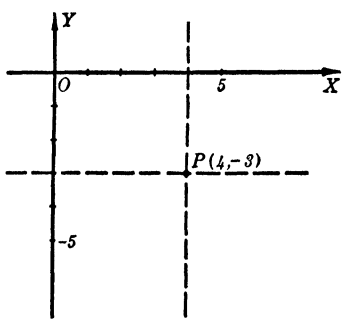


FIG. 2

In practice, it is customary to start at the origin O and measure a distance along the x axis, in this case 4 units to the right, then measure from this point along a perpendicular to the x axis, in this case 3 units downward, and mark the point.

We may then think of the point $(4, -3)$ as being the intersection of two lines, one parallel to the y axis and 4 units to the right of it, the other parallel to the x axis and 3 units below it.

The axes should always be marked so as to show the scale of units used.

3. Theorem for Directed Lines. *If O, any point on the line A_1A_2, is taken as an origin, then the directed line-segment A_1A_2 expressed in terms of OA_1 and OA_2 is always equal to OA_2 minus OA_1.*

Three cases are to be distinguished, one when O is between A_1 and A_2, the others when O is outside the segment A_1A_2. Let O be between A_1 and A_2; then

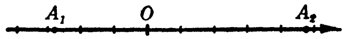


FIG. 3

$$A_1A_2 = A_1O + OA_2.$$

But $A_1O = -OA_1$; hence

$$A_1A_2 = -OA_1 + OA_2 = OA_2 - OA_1,$$

as was to be proved. The other two cases are left as exercises.

4. Horizontal and Vertical Projections of a Line-Segment. Let P_1P_2 be any line-segment whose extremities have the coordinates (x_1, y_1) and (x_2, y_2) respectively. Through each extremity of the segment draw a horizontal and a vertical line. The distance between the vertical lines, measured along a horizontal, that is, AP_2 or M_1M_2, is known as the ***horizontal distance*** from P_1 to P_2, or the ***horizontal projection of P_1P_2***. Similarly the distance between the horizontal lines as measured along a vertical, namely, P_1A or N_1N_2 is known as the ***vertical distance*** from P_1 to P_2, or the ***vertical projection of P_1P_2***.

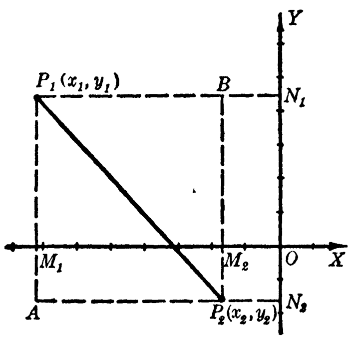


FIG. 4

To express these horizontal and vertical projections in terms of the coordinates of P_1 and P_2, we have

$$M_1M_2 = M_1O + OM_2 = -x_1 + x_2.$$

Hence *the horizontal projection of* P_1P_2 *equals* $x_2 - x_1$. Similarly,

$$N_1N_2 = N_1O + ON_2 = -y_1 + y_2.$$

Hence *the vertical projection of* P_1P_2 *equals* $y_2 - y_1$. That is, the horizontal projection of P_1P_2 is the abscissa of the last-named point minus the abscissa of the first-named point; the vertical projection of P_1P_2 is the ordinate of the last-named point minus the ordinate of the first-named point.

5. Length of a Line-Segment. From Fig. 4, we observe that the horizontal and vertical projections of P_1P_2, namely,

AP_2 and P_1A respectively, are the two legs of the right triangle P_1P_2A. Hence, letting P_1P_2 be represented by d, we have

$$d^2 = \overline{P_1P_2}^2 = \overline{P_1A}^2 + \overline{AP_2}^2.$$

But

$$P_1A = P_1M_1 + M_1A = -y_1 + y_2 = y_2 - y_1,$$

and

$$AP_2 = AN_2 + N_2P_2 = -x_1 + x_2 = x_2 - x_1.$$

Hence

(I) $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLES

1. Given $A(-7, 3)$ and $B(-1, -5)$, find
(*a*) the vertical projection of AB;
(*b*) the horizontal projection of BA;
(*c*) the length of the segment AB.

SOLUTION. (*a*) The vertical projection of AB is $(-5) - (3) = -8$ units.

(*b*) The horizontal projection of BA is $(-7) - (-1) = -6$ units.

(*c*) The length of AB is

$$d = \sqrt{(+6)^2 + (-8)^2} = 10 \text{ units.}$$

2. Given $A(c, d)$ and $B(e, -c)$ find
(*a*) the horizontal projection of AB;
(*b*) the vertical projection of BA;
(*c*) the length of the segment AB.

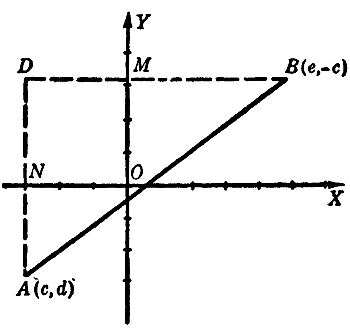


FIG. 5

SOLUTION. (*a*) The horizontal projection of AB is

$$DB = DM + MB = -c + e = e - c.$$

(*b*) The vertical projection of BA is

$$DA = DN + NA = -(-c) + d = d + c.$$

(*c*) The length of AB is

$$\begin{aligned} d &= \sqrt{\overline{AD}^2 + \overline{DB}^2} \\ &= \sqrt{(AN + ND)^2 + (DM + MB)^2} \\ &= \sqrt{(-d - c)^2 + (-c + e)^2} \\ &= \sqrt{(c + d)^2 + (e - c)^2}. \end{aligned}$$

PROBLEMS

1. Prove the theorem for directed lines when
   (*a*) the point O is on AB extended;
   (*b*) the point O is on BA extended.

2. Find the horizontal and the vertical projections of the line-segment P_1P_2; of P_2P_1; also find the length of the segment for $P_1(-4, -4)$ and $P_2(1, 3)$. *Ans.* $5, 7;\ -5, -7;\ \sqrt{74}$ units.

3. The same as Problem 2 for $P_1(0, 1)$, $P_2(-8, 0)$.

4. The same as Problem 2 for $P_1(-3, 3)$, $P_2(3, -3)$.

5. The same as Problem 2 for $P_1(\sqrt{3}, 2)$, $P_2(3, -\sqrt{3})$.

6. The same as Problem 2 for $P_1(a, b)$, $P_2(c, d)$.

7. The same as Problem 2 for $P_1(m, -n)$, $P_2(n, -m)$.
   Ans. $n - m, n - m;\ m - n, m - n;\ (m - n)\sqrt{2}$ units.

8. The same as Problem 2 for $P_1(-g, h)$, $P_2(h, -g)$.

6. Inclination. Slope. *The* ***inclination*** *of a line is the angle which its positive direction makes with the positive direction of the x axis.*

The ***slope*** *of a line is the tangent of its inclination.*

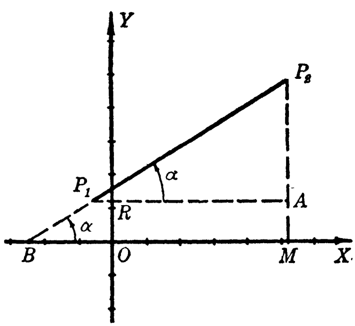


FIG. 6

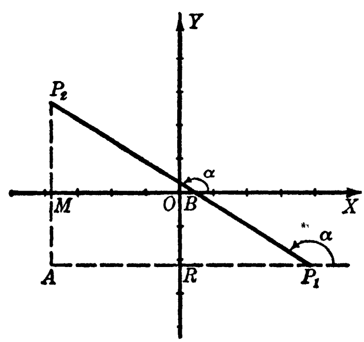


FIG. 7

From either figure we have

$$\tan \alpha = \frac{MP_2}{BM} = \frac{AP_2}{P_1A}.$$

But

$$AP_2 = AM + MP_2 = -y_1 + y_2 = y_2 - y_1,$$

and

$$P_1A = P_1R + RA = -x_1 + x_2 = x_2 - x_1.$$

Hence, letting the slope of P_1P_2 be m, we have

(II) $$\tan \alpha = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

That is, the slope of the line joining two given points is the difference of the ordinates of the two points divided by the difference of their abscissas taken in the same order.

7. Conditions for Parallelism and Perpendicularity. If two lines L_1 and L_2 are parallel, they have the same inclination and their slopes are equal, that is, the condition for two lines with slopes m_1 and m_2 respectively to be parallel is

(III) $$m_1 = m_2.$$

Conversely, if their slopes are equal, $\tan \alpha_1 = \tan \alpha_2$, hence the inclinations are equal and the lines are parallel.

If L_1 and L_2 are perpendicular, their inclinations must differ by 90°. If L_1 has the greater inclination, then

$$\alpha_1 = \alpha_2 + 90°.$$

Hence

$$\tan \alpha_1 = \tan (\alpha_2 + 90°)$$
$$= -\operatorname{ctn} \alpha_2$$
$$= -\frac{1}{\tan \alpha_2}.$$

But

$$\tan \alpha_1 = m_1, \quad \text{and} \quad \tan \alpha_2 = m_2.$$

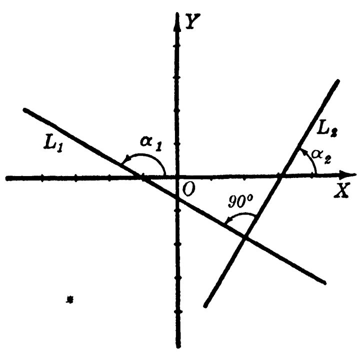


FIG. 8

Therefore

(IV) $$m_1 = -\frac{1}{m_2}, \quad \text{or} \quad m_1m_2 = -1.$$

Conversely, if $m_1m_2 = -1$, then $\tan \alpha_1 = -1/\tan \alpha_2 = -\operatorname{ctn} \alpha_2$. Hence the inclinations of the lines must differ by 90° and the lines are perpendicular. Therefore:

A necessary and sufficient condition for two lines to be perpendicular is that their ***slopes be negative reciprocals.***

8. The Angle between Two Lines. By ***the angle*** between two lines is meant the angle formed by the positive directions from

their intersection, the angle being measured counter-clockwise. This angle is the one which lies entirely above the intersection of the lines.

In Fig. 9, β is the angle between L_2 and L_1, and therefore

(V) $$\beta = \alpha_1 - \alpha_2,$$

or

$$\tan \beta = \tan (\alpha_1 - \alpha_2) = \frac{\tan \alpha_1 - \tan \alpha_2}{1 + \tan \alpha_1 \tan \alpha_2}.$$

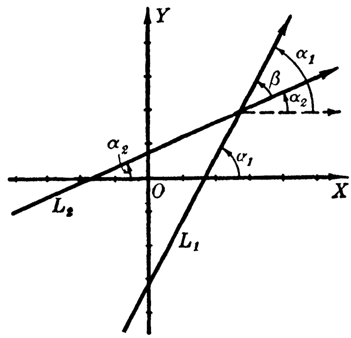


FIG. 9

Hence

(VI) $$\tan \beta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Therefore the angle between two lines is the greater inclination minus the lesser, or the tangent of the angle between two lines in terms of their slopes is $(m_1 - m_2)/(1 + m_1 m_2)$, *where* m_1 ***is the slope of the line with the greater inclination.***

9. Mid-Point of a Line-Segment. The coordinates of the mid-point P of the line-segment joining the points P_1 and P_2 may be expressed in terms of the coordinates of P_1 and P_2 as follows.

Draw the horizontal and vertical lines through P_1, P, and P_2, as shown in Fig. 10. Since $P_1P = PP_2$, we have at once $BP = CP_2$. But

$$BP = BN + NP = -x_1 + x,$$

and

$$CP_2 = CS + SP_2 = -x + x_2.$$

Hence

$$x - x_1 = x_2 - x,$$

or

$$x = \frac{x_1 + x_2}{2}.$$

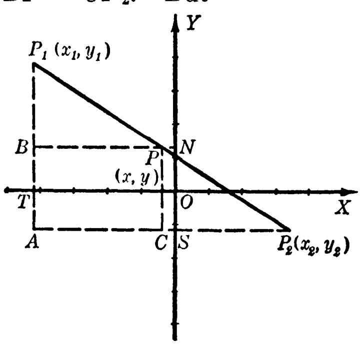


FIG. 10

Similarly, the vertical projection of P_1P is equal to the vertical projection of PP_2; and, by similar reasoning, we find

$$y = \frac{y_1 + y_2}{2}.$$

Therefore, the coordinates of the mid-point of the line-segment P_1P_2 are

$$\text{(VII)} \qquad x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}.$$

EXAMPLES

1. Find the angles of the triangle whose vertices are $A(-7, 6)$, $B(2, -2)$, and $C(-3, -4)$.

SOLUTION. From the triangle ASC, we have the slope of CA given by

$$\frac{SA}{CS} = \frac{SM + MA}{CN + NS} = \frac{4 + 6}{3 - 7} = -\frac{5}{2}.$$

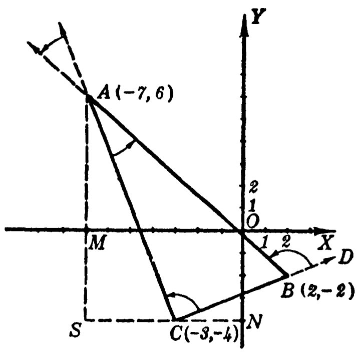


FIG. 11

Similarly, or from the formula for the slope of a line segment, we find that the slope of BA is $-8/9$ and that of CB is $2/5$. Hence, CA and CB are perpendicular, their slopes being negative reciprocals. That is, $\angle BCA$ is a right angle. The angle between CA and BA is equal to $\angle CAB$; therefore

$$\tan \angle CAB = \frac{(-8/9) - (-5/2)}{1 + (-8/9)(-5/2)} = \frac{1}{2}.$$

Hence $\angle CAB = 26° 33.9'$.

The remaining angle of the triangle is the supplement of the angle between the lines BA and CB. Hence formula (VI) gives

$$\tan \angle DBA = \frac{(-8/9) - (2/5)}{1 + (-8/9)(2/5)} = -2.$$

Therefore

$$\tan \angle ABC = \tan (180° - \angle DBA) = -\tan \angle DBA = 2,$$

or

$$\angle ABC = 63° 26.1'.$$

2. Given $A(1, 4)$ and $B(5, -2)$. (*a*) If B is the mid-point of AC, find the coordinates of C. (*b*) If AD is perpendicular to AB and $AD = AB$, find the coordinates of D.

SOLUTIONS. (*a*) Let the coordinates of C be (x_1, y_1), then, since B is the mid-point of AC, we have from the mid-point formulas

$$5 = \frac{x_1 + 1}{2}, \qquad -2 = \frac{y_1 + 4}{2},$$

whence (x_1, y_1) is the point $(9, -8)$.

Using slopes, since the slope of AB is $-3/2$, we see that B is 6 units below and 4 units to the right of A. Accordingly, C is 6 units below and 4 units to the right of B, which locates it at $(9, -8)$.

(*b*) The slope of AD is $2/3$, since it is perpendicular to AB. Then since $AD = AB$, D is either 6 units to the right and 4 units above A, or 6 units to the left and 4 units below A. These locate D at either $(7, 8)$ or $(-5, 0)$.

These results may also be obtained algebraically as follows. Let D have the coordinates (x, y). Then, using the slope of AD, we have

$$\frac{y-4}{x-1} = \frac{2}{3}.$$

Also, since $AD = AB = 2\sqrt{13}$, we have

$$\sqrt{(x-1)^2 + (y-4)^2} = 2\sqrt{13}.$$

Solving these two equations simultaneously, we have

(1) $\quad (x-1)^2 + (y-4)^2 = 52,$

but from the first equation

(2) $\quad y - 4 = \frac{2}{3}(x-1).$

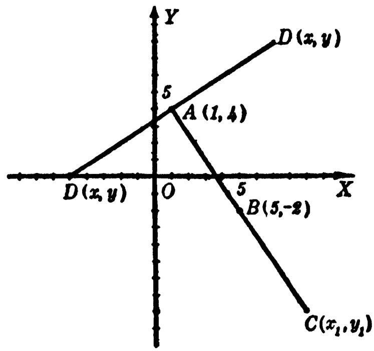


FIG. 12

Substituting this value for $(y - 4)$ in (1), and solving for $(x - 1)$, we get

$$x - 1 = \pm 6, x = 7 \text{ or } -5;$$

whence, by (2), $y = 8$ or 0.

PROBLEMS

1. Given the triangle with vertices $(6, 8)$, $(-4, -2)$, $(8, 4)$

(*a*) Find the length of the sides. *Ans.* $6\sqrt{5}$, $2\sqrt{5}$, $10\sqrt{2}$ units.

(*b*) Find the length of the medians.

(*c*) Find the slope and inclination of each side.

(*d*) Find the angles of the triangle.

Ans. (*c*) $1/2$, $26° 33.9'$; -2, $116° 33.9'$; 1, $45°$.

Ans. (*d*) $71° 33.9'$; $18° 26.1'$; $90°$.

2. The same as Problem 1 for $(-6, -1)$, $(-2, -4)$, $(4, 3)$.

3. Find the distance from $(-4, 6)$ to the mid-point of the segment joining $(7, 1)$ and $(-3, -9)$. *Ans.* $2\sqrt{34}$ units.

4. Do the following sets of three points lie on a line? Prove your answers. (*a*) $(-9, 2)$, $(-2, -1)$, $(11, -7)$; (*b*) $(3, -1)$, $(23, 14)$, $(15, 8)$.

5. Construct a line through $(4, -4)$ with a slope of $-7/5$. Do the points $(-6, 10)$ and $(10, -12)$ lie on this line? Find the points on this line with integral coordinates which are nearest to the point $(4, -4)$.

6. What equation must the coordinates of $P(x, y)$ satisfy if P is 11 units from the point $(7, 2)$?

7. If the point (a, b) is in the second quadrant, where is the point (b, a)? What are the coordinates of the mid-point of the line joining these points, and in what quadrants may it lie?

8. Prove analytically that the lines joining the mid-points of the sides of any triangle are each equal to half the opposite side and parallel to it. [Take the vertices at points $(2a, 0)$, $(2b, 0)$, and $(0, 2c)$.]

9. The mid-points of the sides of a given triangle are $(-2, 3)$, $(4, 1)$, and $(-1, -2)$, respectively. Find the vertices.

Ans. $(5, -4)$, $(-7, 0)$, $(3, 6)$.

10. Find the inclination of each of the following lines. Construct each line by using only one point and its slope:

Through (*a*) $(-2, 0)$ and $(5, -3)$; (*b*) $(1, 3)$ and $(-2, 7)$; (*c*) $(4, -5)$ and $(-3, -6)$.

11. One end of a line-segment 13 units long is $(-4, 8)$ and the ordinate of the other end is 3. What is the abscissa of that end? *Ans.* 8, or -16.

12. The vertices of any quadrilateral are taken at $(2a, 0)$, $(0, 2b)$, $(2c, 0)$ and $(2d, 2e)$. Prove that the lines joining the mid-points of the sides taken in order form a parallelogram. What are the coordinates of the intersection of the diagonals of the parallelogram?

13. Two of the vertices of an equilateral triangle are at $(2, 2\sqrt{3})$ and $(-2, -2\sqrt{3})$. Find the coordinates of the third vertex.

Ans. $(-6, 2\sqrt{3})$ or $(6, -2\sqrt{3})$.

14. The extremities of a diagonal of a square are at $(-5, 2)$ and $(3, -6)$. Find the coordinates of the other vertices.

15. The same as Problem 14, for the points $(-6, 2)$ and $(2, -4)$.

Ans. $(1, 3)$ or $(-5, -5)$.

16. The point $P(x, y)$ is as far from the origin as it is from the point $(4, -6)$. What equation must its coordinates satisfy?

17. A given line has a slope of $2/3$. Find the slope of a line which makes with the given line: (*a*) an angle of 45°; (*b*) an angle of 135°. *Ans.* (*a*) 5.

18. Find the slope of a line which makes an angle of 60° with a line whose slope is $2\sqrt{3}$. How many solutions are there?

10. Graphs. The locus of all points whose coordinates x and y satisfy a given equation is called the ***curve*** or ***graph*** of the equation. If the equation is algebraic, the corresponding graph is an algebraic curve. Other equations and curves, such as exponential and trigonometric, are called ***transcendental.*** The graph of an equation can be approximated by plotting a series of points whose coordinates satisfy the equation, and then drawing a smooth curve through them.

The following procedure will facilitate the work involved:

(*a*) Solve the equation for y in terms of x.

(*b*) Make out a table of values for x and y by assigning positive and negative values for x and calculating for each the corresponding value or values of y.

(*c*) Plot the points designated by x and y as abscissa and ordinate respectively. Then draw a smooth curve through the points.

If it is inconvenient or impossible to solve the equation for y, it may be solved for x in terms of y, in which case arbitrary values are to be assigned to y and corresponding values for x are calculated to make out a table of values.

Certain information about the graph of a given equation can be found which will enable the student to sketch the curve by plotting only a few points rather than by making out a lengthy table of values of x and y. The topics to be considered include ***intercepts, symmetry, extent of curve, horizontal*** and ***vertical asymptotes.***

INTERCEPTS. The ***x intercepts*** are the abscissas of the intersections of the curve with the x axis. They are found by setting y equal to zero in the given equation and solving the result for x. Similarly, the ***y intercepts*** are the ordinates of the intersections of the curve with the y axis. They are found by setting x equal to zero in the given equation and solving the result for y.

EXAMPLE

Find the intercepts of $x^2 - 2y^2 - 4x - 5 = 0$.

SOLUTION. Setting $y = 0$ in the equation we have

$$x^2 - 4x - 5 = 0, \qquad x = 5 \text{ or } -1.$$

Similarly for $x = 0$, we get $y = \pm\sqrt{-2.5}$. Hence the x intercepts are 5 and -1; the y-intercepts are imaginary, in other words *the curve does not cross the y axis*.

SYMMETRY. Two points are symmetric with respect to the x axis when they have the same abscissa and their ordinates differ only in sign. If they are symmetric with respect to the y axis, they have the same ordinate and their abscissas differ only in sign. Two points are symmetric with respect to the origin as a center when their respective coordinates are numerically equal but of opposite signs.

If a curve is symmetric with respect to the x axis, each point on the curve has the point symmetric to it with respect to the x axis also on the curve. Then if (x, y) represents any point on

the curve, the point $(x, -y)$ must also be on the curve. That is, if x and y represent any pair of numbers which satisfy the equation of the curve, that equation will be satisfied by x and $-y$ also. We have then the following test:

If in the given equation $-y$ *can be substituted for* $+y$ *without changing the equation, then the graph is symmetric with respect to the* x *axis.*

Obviously this condition is satisfied if y is involved in the equation ***only*** with ***even powers.***

Similarly, *if in a given equation* $-x$ *can be substituted for* $+x$ *without changing the equation, then the graph of the equation is symmetric with respect to the* y *axis.*

If the substitution of both $-x$ *for* $+x$ *and* $-y$ *for* $+y$ *in a given equation does not change the equation, then its graph is symmetric with respect to the origin.*

EXTENT OF THE CURVE. In general, it is easy to determine whether the graph of a given equation is a closed or an open curve, and whether there is any region of the plane between two horizontal lines, or between two vertical lines, in which the curve does not exist. To do so, solve the equation for y in terms of x; if then this value of y involves a square root or an even root such that all values of x between, say, $x = a$ and $x = b$ make y imaginary, then no part of the curve can be between the vertical lines $x = a$ and $x = b$. On the other hand, if values of x between a and b are the only ones which make y real, then the curve lies wholly between these two vertical lines. If y is real for all values of x, the curve is unlimited in its extent along the x axis.

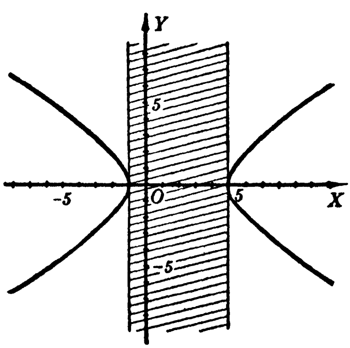


FIG. 13

Next solve the equation for x in terms of y and find in a similar manner whether or not any values of y make x imaginary, and whether or not the curve is restricted or unlimited in extent in the direction of the y axis.

EXAMPLE

Consider the equation

$$x^2 - 2y^2 - 4x - 5 = 0$$

mentioned above. Its graph is symmetric with respect to the x axis, but not with respect to the y axis, nor the origin. Why?

SOLUTION. Solving the equation for each variable we have

$$y = \pm\sqrt{(1/2)(x-5)(x+1)}, \qquad x = 2 \pm \sqrt{2\,y^2+9}.$$

From the first of these we observe that all values of x between $x = -1$ and $x = +5$ make y imaginary, while all values of x greater than or equal to 5 and less than or equal to -1 make y real. From the second, all values of y give two real values of x. Also as x increases beyond 5 or decreases beyond -1, y^2 increases and the curve extends indefinitely, as indicated in Fig. 13.

PROBLEMS

Discuss and plot the graph of each of the following equations. (Nos. 1–20.)

1. $x^2 = 4\,y.$
2. $y^2 + 4\,x + 12 = 0.$
3. $3\,x = y^2 - 6.$
4. $x^2 + 6\,y - 15 = 0.$
5. $x^2 + y^2 - 6\,x + 4\,y = 0.$
6. $x^2 + y^2 - 6\,y = 16.$
7. $4\,x^2 + y^2 = 24.$
8. $x^2 + 4\,y^2 = 16.$
9. $3\,x^2 - y^2 = 12.$
10. $x^2 - 3\,y^2 + 12 = 0.$
11. $9\,x^2 - 4\,y^2 = 0.$
12. $x^2 + 3\,y^2 - 15\,y = 0.$
13. $xy = 12.$
14. $xy^2 = 12.$
15. $x^3 + 4\,y^2 = 0.$
16. $y = 4\,x - x^3.$
17. $y^2 = x^3 - 3\,x^2.$
18. $y = x - 1/x.$
19. $y(x^2 + 4) = 8.$
20. $y = x^2(x - 2).$

Discuss the graph of each of the following equations, and plot by giving appropriate lengths to the literal coefficients. (Nos. 21–28.)

21. $y^2 = 2\,px.$
22. $xy = 2\,a.$
23. $x^2/a^2 + y^2/b^2 = 1.$
24. $x^2/a^2 - y^2/b^2 + 1 = 0.$
25. $x^{1/2} + y^{1/2} = a^{1/2}.$
26. $x^{2/3} + y^{2/3} = a^{2/3}.$
27. $y = 8\,a^3/(x^2 + 4\,a^2).$
28. $y^2 = ax^3.$

11. Asymptotes. An ***asymptote*** of a curve is a straight line which the curve approaches continuously in such a way that the distance between the line and the curve approaches zero as they are indefinitely extended. If a curve has either horizontal or vertical asymptotes they are easily found when the equation is solved for x and y. Thus if y in terms of x is an algebraic fraction and any value of x such as $x = a$ makes the denominator zero and the numerator different from zero, then y increases indefinitely as x approaches a and $x = a$ is a ***vertical asymptote.*** Similarly, to find horizontal asymptotes, solve for x. If the result is a fraction

whose denominator becomes zero for some value of y such as $y = b$ when the numerator differs from zero, then $y = b$ is a ***horizontal asymptote.***

EXAMPLE

Trace the curve $x^2y - x^2 - 2y + 4 = 0$.

SOLUTION. The intercepts on the x axis are $+2$, -2, on the y axis 2. The graph is symmetric with respect to the y axis. Solving for x and y, we have

$$x = \pm\sqrt{\frac{2(y-2)}{y-1}}, \qquad y = \frac{x^2-4}{x^2-2}.$$

If y is between 1 and 2, x is imaginary; hence no part of the curve is between the lines $y = 1$ and $y = 2$. The horizontal asymptote is $y = 1$; the vertical are $x = \pm\sqrt{2}$. This information with a few points on each branch give the curve as shown in Fig. 14.

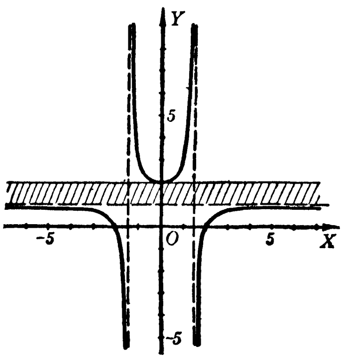


FIG. 14

12. Intersections of Curves. From the definition of the graph of an equation, it follows that if two curves intersect, the coordinates of each point of intersection must satisfy both equations. Hence, *to find the coordinates of all points of intersection of two curves, solve the corresponding equations simultaneously.*

In solving two equations simultaneously only real values for both x and y will give a real point of intersection, so imaginary or complex values may be disregarded.

EXAMPLE

Find the intersections of the curves whose equations are $9x^2 + 4y^2 = 37$, $y = x^2 - 1$.

SOLUTION. Eliminating x^2 between the two equations, we have

$$4y^2 + 9(y+1) = 37,$$

or

$$4y^2 + 9y - 28 = 0,$$

whence

$$(y+4)(4y-7) = 0.$$

Therefore

$$y = -4, \frac{7}{4}.$$

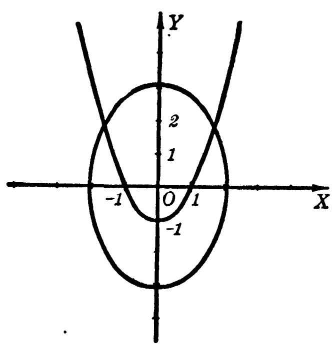


FIG. 15

For $y = -4$, $x = \pm\sqrt{-3}$ and there are no intersections; for $y = 7/4$,

$x = \pm\sqrt{11/4} = \pm 1.66$. Hence the intersections are (1.66, 1.75) and (− 1.66, 1.75). The solution should be checked by drawing the graphs.

PROBLEMS

Discuss and draw the asymptotes, then plot the graph of each of the following equations. (Nos. 1–16.)

1. $xy + 2x = 4$.
2. $y = 2x + xy$.
3. $y = 2x + xy + 2$.
4. $y = 3x + xy + 4$.
5. $xy + 3x + y - 6 = 0$.
6. $2xy + 4x - 3y + 6 = 0$.
7. $xy^2 = 4 - x$.
8. $xy^2 - 4x = 12$.
9. $xy^2 - 2y = 4$.
10. $x^2y = 3 + 3y$.
11. $x^2y + 3x^2 = 6$.
12. $x^2y - 2y = x^2 + 4$.
13. $xy^2 = x + 2y^2$.
14. $(y + 4)(x - 1)^2 = 1$.
15. $xy^2 + y^2 - 4x - 2y = 0$.
16. $xy^2 - 2y^2 - 4x = 4$.
17. Find the intersection of $3x - 8y = 20$ and $2x - 7y = 10$. *Ans.* (12, 2).
18. Find the intersections of $(x + 1)^2 + (y + 1)^2 = 13$ with each of the coördinate axes.
19. Find the intersection of $13x + 3y = 9$ and $14x - 4y = 35$. *Ans.* $(1\frac{1}{2}, -3\frac{1}{2})$.

Find the intersections of each of the following pairs of curves and check by drawing their graphs. (Nos. 20–26.)

20. $2x + y = 1$, $y^2 + 4x = 17$.
21. $4x^2 + y^2 = 25$, $8x + 3y + 25 = 0$. *Ans.* (− 2, − 3).
22. $2y = 12 + x$, $x^2 + 4y = 19$.
23. $x^2 - y^2 + 4 = 0$, $x^2 - y = 8$. *Ans.* $(\pm\sqrt{12}, 4)$, $(\pm\sqrt{5}, -3)$.
24. $xy + 8 = 0$, $y^2 = 4x$.
25. $x^2 = 4y$, $y = 8/(x^2 + 4)$. *Ans.* (± 2, 1).
26. $x^2 + 4y^2 + 6x = 0$, $2x^2 - y^2 = 12$.

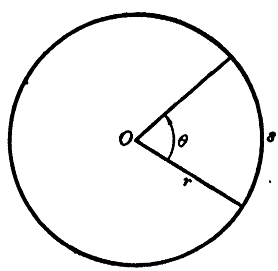


FIG. 16

13. Trigonometric Functions, Circular Measure. If the independent variable x is an angle, it is desirable to have a natural unit of measure to represent its magnitude. This is obtained by measuring the angle in radians. Describe an arc of a circle with its center at the vertex of the angle. *The central angle whose intercepted arc is equal in length to the radius is one* ***radian***. Hence for any central angle the length of the intercepted arc divided by the radius gives the radian measure of the angle.

That is, if s is the length of the arc and r the radius, then

$$\theta = \frac{s}{r} \text{ radians.}$$

Since the circumference is $2\pi r$,

$$180° = \pi \text{ radians,} \qquad 90° = \pi/2 \text{ radians,}$$
$$1° = \pi/180 \text{ radians} = 0.01745 \cdots \text{ radians.}$$
$$1 \text{ radian} = 180/\pi \text{ degrees} = 57° \, 17.7'.$$

To plot the graph of trigonometric functions, make out a table of values of the angle in radians and of the required function. Thus:

x	$\sin x$	$\tan x$	$\sec x$
0	0.00	0.00	1.00
$\pi/6 = 0.52$	0.50	0.58	1.15
$\pi/3 = 1.05$	0.87	1.73	2.00
$\pi/2 = 1.57$	1.00	∞	∞
$2\pi/3 = 2.09$	0.87	−1.73	−2.00
$5\pi/6 = 2.62$	0.50	−0.58	−1.15
$\pi = 3.14$	0.00	0.00	−1.00

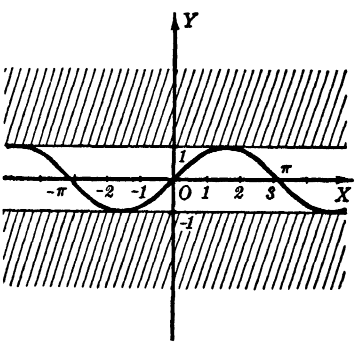


FIG. 17. $y = \sin x$

The graph of

$$y = \sin x$$

will cross the x axis for each value of x equal to an integral multiple of π. Moreover, since $\sin(-x) = -\sin x = -y$ the graph of $y = \sin x$ is symmetric with respect to the origin. The same is true for $y = \tan x$. But since $\sec(-x) = \sec x$ the graph of $y = \sec x$ is symmetric with respect to the y axis. These curves are given below. Likewise, asymptotes of the graphs of $\tan x$ and $\sec x$ are shown.

A function whose values are repeated after a definite interval p of the variable, so that

$$f(x + p) = f(x)$$

is called a ***periodic function.*** The interval p is called the ***period.*** The maximum absolute value of the function is called the ***amplitude.*** Thus, in the function $\sin x$, the period is 2π and the amplitude is 1. The period of $\tan x$ is $p = \pi$ and it does not have a finite amplitude. The period of the secant is 2π.

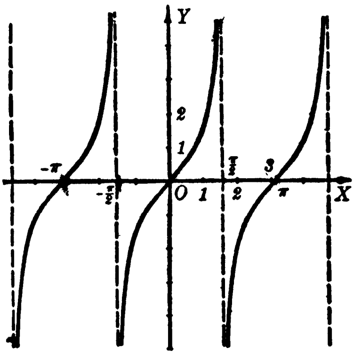


FIG. 18. $y = \tan x$

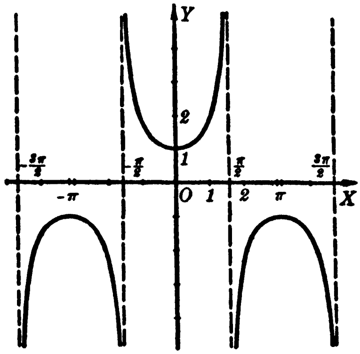


FIG. 19. $y = \sec x$

If we consider the function

$$y = k \sin nx,$$

it is evident that $\sin nx$ will be zero when nx is zero or any integral multiple of π, that is, when x is 0, π/n, $2\pi/n$, etc. Again, the period of this function is $2\pi/n$. The maximum value of $\sin nx$ is $+1$ when nx is $\pi/2$ or differs from $\pi/2$ by a multiple of 2π, that is to say, when x is $\pi/2n$ or differs from $\pi/2n$ by an integral multiple of the period $2\pi/n$. Hence the amplitude of the function $k \sin nx$ is k.

14. Inverse Trigonometric Functions. The function

$$y = \sin^{-1} x$$

means in direct notation

$$x = \sin y.$$

The function ***sin⁻¹ x*** is called *the inverse of the function sin x,* and is read ***the angle whose sine is x,*** or ***arc sine x.*** Hence the graph of $y = \sin^{-1} x$ differs from that of $y = \sin x$ only by having the coordinate axes interchanged. The same is true for the other inverse trigonometric functions.

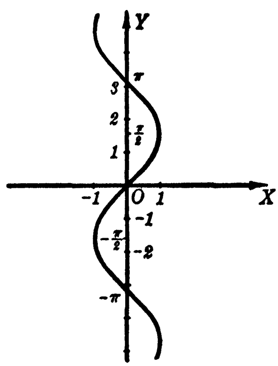


FIG. 20. $y = \sin^{-1} x$

15. The Algebraic Sum of Functions. If the given function consists of a sum of functions, such as

$$y = f_1(x) + f_2(x),$$

the following alternative method may be used to find the graph of the function. Plot on the same axes the graphs of $y = f_1(x)$ and $y = f_2(x)$. Along each vertical line of the coordinate paper mark the point whose ordinate is the sum of the ordinates of $f_1(x)$ and $f_2(x)$. The desired graph is the smooth curve through the points located in this manner. Of course, this method can only be used for values of x which make both $f_1(x)$ and $f_2(x)$ real.

EXAMPLE

Draw the graph of $y = (3/2) \cos x + \sin (2x/3)$.

SOLUTION. Plot with the same axes the graph of $y = (3/2) \cos x$ and the graph of $y = \sin (2x/3)$. The addition of corresponding ordinates gives the required graph, as shown in Fig. 21.

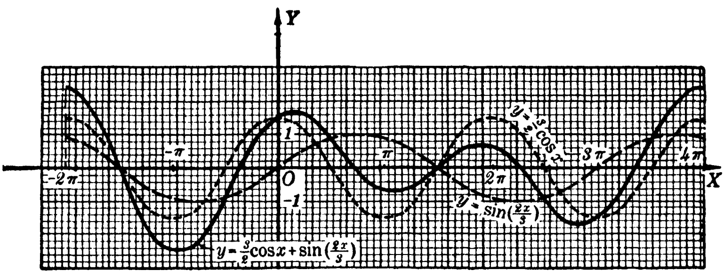


FIG. 21

PROBLEMS

Plot the graph of each of the following and find its period. (Nos. 1–9.)

1. $y = \cos x$.
2. $y = \text{ctn } x$.
3. $y = \csc x$.
4. $y = 2 \cos (x/2)$.
5. $y = 2 \sin (\pi x/3)$.
6. $y = \tan (x + \pi/4)$.
7. $y = \sin x + \cos x$.
8. $y = \cos x - \sin (x/2)$.
9. $y = (1/2) \text{ ctn } (\pi x/4)$.

Plot the graph of each of the following and tell which are periodic.

10. $y = (1/2) \sin^{-1} x$.
11. $y = 2 \cos^{-1} 2x$.
12. $2y = \tan (x/2 - \pi/4)$.
13. $x = 2 \text{ ctn}^{-1} 2y$.
14. $y = 3 \sin 2x$.
15. $y = 3 \sin^{-1} 2x$.
16. $y = 1 + \cos (\pi x/2)$.

17. $y = x + 2 \sin x, \ -2\pi \leqq x \leqq 2\pi$.

18. $y = x \sin x, \ -2\pi \leqq x \leqq 2\pi$.

19. $x = 2 \sin^{-1} (y - 2)$.

20. $y = \cos^{-1} (x + 1), \ -\pi \leqq y \leqq \pi$.

21. $y = 2 \cos x - \cos 2x, \ 0 \leqq x \leqq 2\pi$.

22. $y = \cos x - 2 \sin (x/2), \ 0 \leqq x \leqq 4\pi$.

23. $x = \pi/2 + \tan^{-1} (y/2)$.

24. $y = \sin (\pi x/2) - \cos (2\pi x/3)$, for 1 period.

25. $y = x \cos (\pi x/2)$.

26. $y = x^2 \cos \pi x$.

16. Exponential and Logarithmic Curves. The equation $y = a^x$, in which a is any constant, is an ***exponential equation.*** If a is positive, y is real for all real values of x and a table of values can easily be obtained from a table of logarithms by the relation $\log y = x \log a$. If a is negative, y is not real for all values of x between any two given values. Thus, if x is given a fractional value whose denominator is an even number, y would be an even root of a negative number and hence imaginary. Hence the graph of $y = a^x$ is a smooth unbroken curve only if a is positive. We shall consider only such equations. Since $(1/b)^x = (b)^{-x}$, if the base is less than one, the equation can be written with the reciprocal base. Hence we assume throughout that a is positive and greater than unity.

The number represented by e ($= 2.71828 \cdots$), the base of ***natural logarithms***, is of great importance in exponential equations. We shall use the symbol *log* instead of $\log_e$ hereafter. Thus if $y = e^x$, $\log y = x$.

EXAMPLE

Draw to the same set of axes the graph of $y = a^x$ when a has each of the values 1.5, e, 4, and $1/e$.

SOLUTION. Form the table shown below.

x	$(1.5)^x$	e^x	4^x	e^{-x}
−2	0.44	0.135	0.06	7.39
−1	0.67	0.37	0.25	2.72
0	1.00	1.00	1.00	1.00
1	1.50	2.72	4.00	0.37
2	2.25	7.39	16.00	0.135

The student should observe carefully the nature of the graph of $y = a^x$ as a takes values that increase from $a = 1$. Also note that the graphs of $y = (1/4)^x = 4^{-x}$, $y = (1/e)^x = e^{-x}$, $y = (2/3)^x = (1.5)^{-x}$ can be obtained from the graphs of $y = 4^x$, $y = e^x$, $y = (1.5)^x$ respectively by a rotation of these graphs through 180° about the y axis. Or, we can say that $y = a^x$ and $y = a^{-x}$ are each symmetric to the other with respect to the y axis; as either can be obtained from the other by changing x to $-x$.

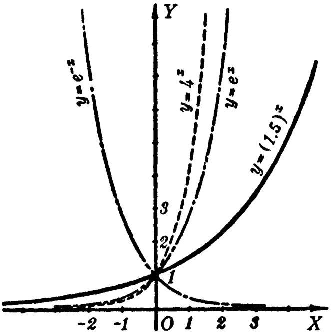


FIG. 22

MULTIPLICATION OF ORDINATES. The graphs of certain equations may be obtained by the multiplication of the corresponding ordinates of auxiliary graphs. Thus if the graphs of $y = f_1(x)$ and $y = f_2(x)$ be drawn to the same set of axes, then the graph of $y = f_1(x) \cdot f_2(x)$ can be obtained for all values of x which make both $f_1(x)$ and $f_2(x)$ real by multiplying the corresponding ordinates of the two auxiliary graphs.

The equation

$$y = ae^{-x} \sin kx$$

plays a very important role in physics. Its graph is known as the ***curve of damped vibration*** and can be drawn by multiplying the ordinates of the graphs of $y = ae^{-x}$ and $y = \sin kx$.

EXAMPLE

Draw an accurate graph of $y = (3/2)e^{-x} \sin (\pi x/2)$.

SOLUTION. Draw the graph of $y = \pm (3/2)e^{-x}$ and also of $y = \sin (\pi x/2)$ to the same axes. Along each ordinate mark the ordinate corresponding to the product of the two given ordinates and draw a smooth curve through the points located in this way. The dotted curves of the figure are called auxiliary graphs and the other curve is the desired graph. Note also that the two curves $y = \pm (3/2)e^{-x}$ are *boundary curves* which the desired curve touches but never crosses.

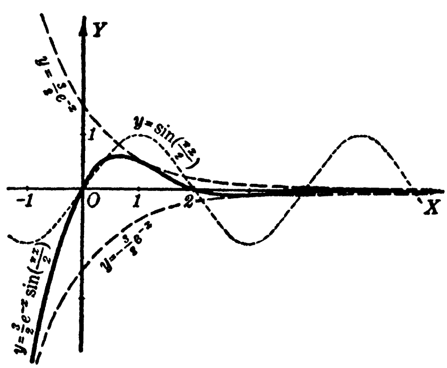


FIG. 23

The logarithmic equation is called the inverse of the exponential since $y = a^x$ may be written $x = \log_a y$. Therefore, if we write $y = \log_a x$, where a is any positive number except unity, its graph is the same as that of $x = a^y$.

EXAMPLES

1. Draw the graph of $y = \log_{10} x$ and also of $y = \log x$ to the same set of axes.

SOLUTION. Form the table shown below.

x	0	0.5	1	2	3	4	5
$\log_{10} x$ $\log x$	$-\infty$ $-\infty$	$-0\ 30$ $-0\ 69$	0 0	0.30 0.69	0.48 1.10	0.60 1.39	0.70 1.61

Notice that the y axis is an asymptote of each curve.

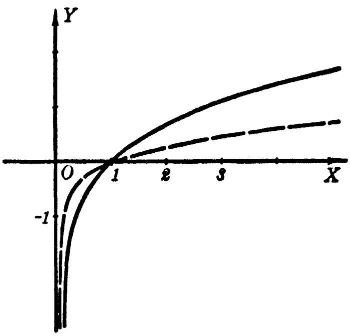


FIG. 24

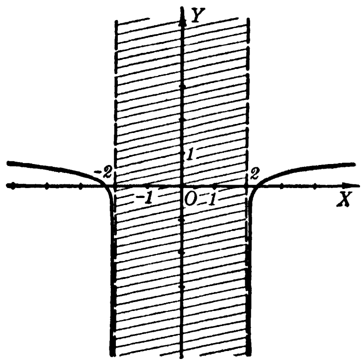


FIG. 25

2. Draw the graph of $y = \log \sqrt{x^2 - 4}$.

SOLUTION. The equation may be written

$$y = (1/2) \log (x^2 - 4).$$

The lines $x = \pm 2$ are asymptotes and the curve crosses the x axis at $x = \pm \sqrt{5}$. If $x^2 < 4$, then y is the logarithm of a negative number, which is complex. Hence no part of the curve lies between the asymptotes. A few points with the information above give us the graph of Fig. 25.

This equation may be written

$$y = (1/2) [\log (x + 2) + \log (x - 2)].$$

However, the student cannot find the graph by the method of the addition of ordinates. If we plot $y = (1/2) \log (x + 2)$ and $y = (1/2) \log (x - 2)$, only the values of x which make both $x + 2$ and $x - 2$ positive can be used in this way. That is, $\log (x - 2)$ is complex between $x = 2$ and $x = -2$ and hence the sum of the ordinates is complex; then for $x < -2$ both

log $(x + 2)$ and log $(x - 2)$ are complex while the sum of these complex numbers is real. Hence this method would give only the part of the curve for which $x > 2$. The auxiliary curves are shown in Fig. 26.

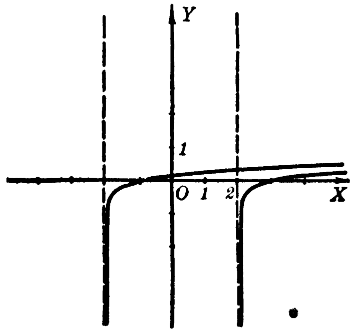


FIG. 26

PROBLEMS

Plot the graph of each of the following equations.

1. $y = (1/2)\, e^{2x}$.
2. $y = (1/2)^{-x}$.
3. $y = (1/2)\, e^{x^2/2}, \ -3 \leqq x \leqq 3$.
4. $y = e^{1-x}$.
5. $y = e^{1/(1-x)}$.
6. $y = 2^{1/(1-x)}$.
7. $y = \log_{10}(3 - x)$.
8. $y = \log (3 - x), \ -3 \leqq x \leqq 3$.
9. $y = \log (3 + x)$.
10. $y = \log \sqrt{9 - x^2}$.
11. $y = \log \sqrt{x^2 - 9}$.
12. $y = \log \sin 2x$.
13. $y = \log (x^2 + 4)$.
14. $y = \log [1/(x^2 + 4)]$.
15. $y = (e^x + e^{-x})/2$, *(Hyperbolic cosine.)*
16. $y = (e^x - e^{-x})/2$, *(Hyperbolic sine.)*
17. $y = (1/3)^{1/x}, \ -5 \leqq x \leqq 5$.
18. $y = e^{-x^2}, \ -2 \leqq x \leqq 2$.
19. $y = xe^x/5, \ -2 \leqq x \leqq 2$.
20. $y = 4\, e^{-x} \sin x$.
21. $y = x^{-1}\, e^{x/2}, \ -1 \leqq x \leqq 3$.
22. $y = e^{-x} - 2 \cos x$.
23. $y = e^{-x/2} \sin \pi x, \ -2 \leqq x \leqq 2$.
24. $y = e^{-x^2} \sin x, \ -3 \leqq x \leqq 3$.
25. $y = 2^{x/3} \cos x, \ 0 \leqq x \leqq 4$.
26. $y = 2^{x/2} \sin 3x, \ -2 \leqq x \leqq 2$.
27. $y = e^{-x/3} \sin 2x, \ -2 \leqq x \leqq \pi$.
28. $y = e^{-x/4} \sin \pi x, \ -3 \leqq x \leqq 2$.
29. $y = e^{x/2} \cos 2\pi x, \ -1 \leqq x \leqq 2$.
30. $y = -3\, e^{-x} \sin \pi x, \ 0 \leqq x \leqq 4$.
31. $y = 2 \cdot 3^{-x/2} \sin 3\pi x, \ -1 \leqq x \leqq 2$.

17. Polar Coordinates. In a system of ***polar coordinates*** the position of a point is determined by measuring a *distance* and a *direction* instead of measuring two distances as in rectangular

coordinates. The system consists of a fixed point O, called the ***pole***, and a fixed line through O called the ***polar axis*** or ***initial line.***

The polar coordinates of any point P in the plane are two numbers r and θ; r represents the distance OP and is called the ***radius vector*** of P, and θ is the angle which OP makes with the polar axis and is called the ***vectorial angle.*** Each coordinate may be positive or negative; θ is positive when measured from the polar axis in a counter-clockwise direction, negative if clockwise; r is positive if OP is along the terminal side of θ, and negative if OP is taken in the opposite direction, along the terminal side produced through the pole.

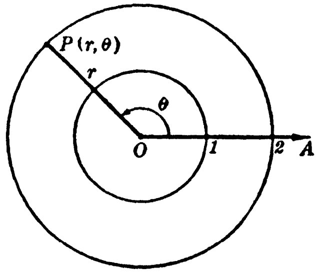


FIG. 27

Any pair of real numbers (r, θ) will determine uniquely a point P. However, the point P may be designated by more than one pair of coordinates. Thus the pairs $(2, 3\pi/4)$, $(2, -5\pi/4)$, and $(-2, -\pi/4)$ each designate the same point.

To *plot points* in polar coordinates, the paper should be ruled with concentric circles about the pole and radial lines through the pole, as shown in Fig. 28.

If an equation is given in r and θ, it can frequently be solved for r in terms of θ. Then by making out a table of corresponding values the graph of the equation is obtained by plotting these points and drawing a smooth curve through them. The student should not assume that the complete graph is obtained when the curve crosses itself. If in doubt, he should continue the table of values until successive points are repeated.

EXAMPLES

1. Plot the graph of $r = 2 - 3 \sin \theta$.

SOLUTION. Giving θ values varying at intervals of 30°, we have:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
r	2	0.5	−0.6	−1	−0.6	0 5	2	3.5	4.6	5.0	4.6	3.5	2
Point	A	B	C	D	E	F	G	H	I	J	K	L	A

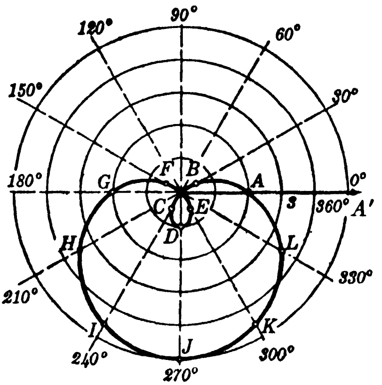


FIG. 28

The complete graph is given by an interval of variation of θ from 0° to 360°, since θ and $(\theta + 2\pi)$ give the same value of r and the same vectorial angle. Between $\theta = \sin^{-1}(2/3)$ in the first and second quadrants r is negative. This curve is one form of the *limaçon*.

2. Plot the graph of

$$r^2 = a^2 \cos 2\theta.$$

SOLUTION. Give θ values at intervals of 15°. If θ is between 45° and 135°, $\cos 2\theta$ is negative and r is imaginary.

θ	0°	15°	30°	45°	135°	150°	165°	180°
r	$\pm a$	$\pm 0.93\,a$	$\pm 0\ 71\,a$	0	0	$\pm 0\ 71\,a$	$\pm 0\ 93\,a$	$\pm a$
Point	A	B	C	O	O	D	E	A

This curve is called the *lemniscate*. It is completely given by an interval of variation of θ from 0° to 180°. This is true since θ and $(\theta + \pi)$ each give the same value for r^2, and also since $(\pm r, \theta)$ and $(\pm r, \theta + \pi)$ locate the same two points. (See Fig. 29.)

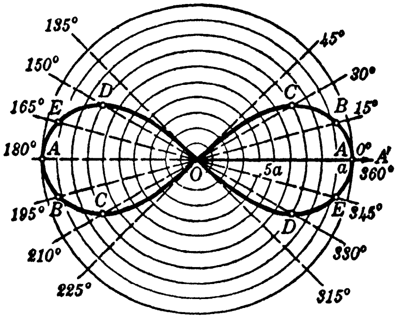


FIG. 29

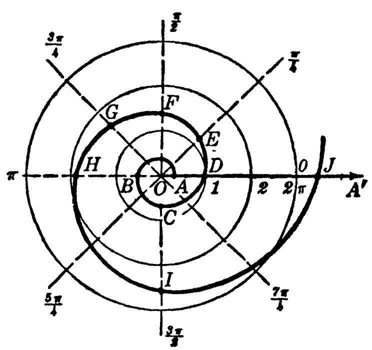


FIG. 30

3. Plot the graph of $r = e^{a\theta}$.

SOLUTION. Let $a = 0.2$ units and express $a\theta$ in radians.

θ	-2π	$-\pi$	$-\pi/2$	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$3\pi/2$	2π
$a\theta$	-1.26	-0.63	-0.31	0	0.16	0.31	0.47	0.63	0.94	1.26
$e^{a\theta}$	0.3	0.5	0.7	1	1.2	1.4	1.6	1.9	2 6	3 5
Point	A	B	C	D	E	F	G	H	I	J

This curve is called the *logarithmic spiral*. (See Fig. 30.)

18. Relations between the Coordinate Systems. At times it is convenient to transform an equation given in rectangular coordinates into a corresponding polar form, and conversely. The usual arrangement is to take the pole at the origin of the rectangular coordinates and the polar axis along the positive half of the x axis. It is apparent that for any point P in the plane the following relations will then exist between its two sets of coordinates (x, y) and (r, θ) :

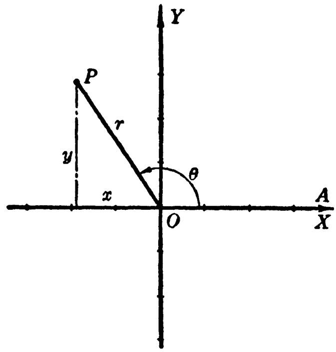


FIG. 31

(VIII) $\quad x = r\cos\theta, \qquad y = r\sin\theta;$

$$\text{(IX)} \quad \begin{cases} r^2 = x^2 + y^2, \\ \theta = \tan^{-1}\dfrac{y}{x} \\ \quad = \cos^{-1}\dfrac{x}{\sqrt{x^2+y^2}}. \end{cases}$$

EXAMPLE

Transform into rectangular coordinates the equation of the lemniscate $r^2 = a^2 \cos 2\theta$.

SOLUTION. Write $\cos 2\theta = \cos^2\theta - \sin^2\theta$, then the equation becomes $r^2 = a^2(\cos^2\theta - \sin^2\theta)$. Therefore, if the axes are as shown above,

$$x^2 + y^2 = a^2\frac{x^2 - y^2}{x^2 + y^2}, \qquad \text{or} \qquad (x^2 + y^2)^2 = a^2(x^2 - y^2).$$

PROBLEMS

Plot the graph of each of the following equations. (Nos. 1–27.)

1. $r\cos\theta = 3.$
2. $r\sin\theta = -2.$
3. $r = -4\sec\theta.$
4. $r = 5\csc\theta.$
5. $r + 2\sin\theta = 0.$
6. $r = 4\cos\theta.$
7. $r = 1/(1 + \cos\theta).$
8. $r = 4/(3 - 2\cos\theta).$
9. $r = 6/(2 - 3\sin\theta).$
10. $2r\cos\theta + r\sin\theta = 3.$
11. $r = 2(1 - \sin\theta).$
12. $r = 3(1 + \cos\theta).$
13. $r = 2 + 3\cos\theta.$
14. $r = 3 - 2\sin\theta.$
15. $r = 1 + \cos 2\theta.$
16. $r = 1 - 2\sin\theta.$
17. $r^2 = 4\cos 2\theta.$
18. $r = 1 + 2\cos(\theta/2).$
19. $r = 1 + \sin(\theta/2).$
20. $r^2 = a^2\sin 2\theta.$

21. $r = a/(1 - e \cos \theta)$, $(e < 1, = 1, > 1)$.

22. $r\theta = a$.

23. $r = a \cos 2\theta$.

24. $r = a \sin 3\theta$.

25. $r = a \sec^2 (\theta/2)$.

26. $r = 2^\theta$.

27. $r = 2 \cos \theta - 3 \sin \theta$.

28. Transform into rectangular coordinates the equations of Problems 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 16.

29. Transform the following equations into polar representation.

(a) $xy = 2a$, (c) $x^2 + y^2 + 2ay = 0$, (e) $x^2(y^2 - a^2) + y^4 = 0$,

(b) $x^2 = 2py$, (d) $x^2 + ax + by = 0$, (f) $x^3 + y^3 - 3axy = 0$.

19. Parametric Equations. We have seen that the equation of a curve may be given by a single relation connecting the coordinates x and y of any point of the curve. Also x and y may each be expressed in terms of a third variable, which is called a ***parameter***, and the two equations together are called the ***parametric equations*** of the curve. The parameter may or may not have geometric significance. Each value arbitrarily assigned to the parameter fixes one or more values for x and y, giving thereby corresponding points on the curve. If the parametric equations of a curve are given, the equation in x and y may be obtained provided the parameter can be eliminated between the given equations.

EXAMPLES

1. Given $x = at^3$, $y = at^2$. Plot the curve and find its equation in terms of x and y.

SOLUTION. Form the table below and plot the points. A smooth curve is obtained by joining the points in order, as shown in Fig. 32.

t	x	y	Point
0	0	0	O
$\pm 1/2$	$\pm a/8$	$a/4$	B
± 1	$\pm a$	a	C
± 2	$\pm 8a$	$4a$	D

Eliminating t, we have $t = \pm\sqrt{y/a}$; then $x = \pm a(y/a)^{3/2}$ or $y^3 = ax^2$.

2. Given $x = a \cos \theta$, $y = b \sin \theta$, obtain the graph and the equation in rectangular coordinates.

SOLUTION. In this example the parameter is the angle θ. The table below gives the coordinates of points on the curve in Fig. 33.

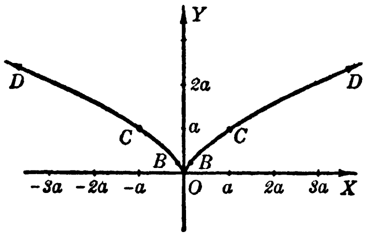


FIG. 32

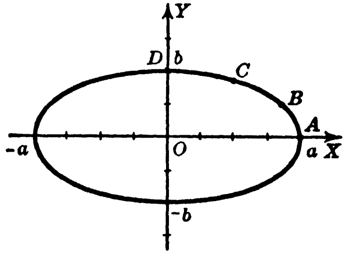


FIG. 33

θ	x	y	Point
0°	a	0	A
30°	$a\sqrt{3}/2$	$b/2$	B
60°	$a/2$	$b\sqrt{3}/2$	C
90°	0	b	D
.	.	.	.
.	.	.	.
.	.	.	.

To eliminate θ we have $\cos\theta = x/a$, $\sin\theta = y/b$. Using the relation $\sin^2\theta + \cos^2\theta = 1$, we find the equation to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

PROBLEMS

Make a table of values and plot the graph of each of the following pairs of equations. (Nos. 1–10.)

1. $x = 1 - t,\ y = 4t.$

2. $x = 1 - t^2,\ y = 4t.$

3. $x = 2/t,\ y = 3t.$

4. $x = 1 + 2t,\ y = 6/t.$

5. $x = a\sin t,\ y = a\cos t.$

6. $x = 4\cos\theta,\ y = 3\sin\theta.$

7. $x = 4\sec\theta,\ y = 6\tan\theta.$

8. $x = a\cos^3\theta,\ y = a\sin^3\theta.$

9. $x = 3/(t-1),\ y = 2/(t+1).$

10. $x = 3t/(t^3+1),\ y = 3t^2/(t^3+1).$

11. Derive the rectangular equations of Problems 1 to 10 inclusive by eliminating the parameter.

Ans. (1) $4x + y = 4$; (3) $xy = 6$; (6) $x^2/16 + y^2/9 = 1$; (10) $x^3 + y^3 = 3xy$.

20. Transformation of Coordinates. It is sometimes desirable to simplify the equation of a given locus by referring it to a new set of coordinate axes, since the form of the equation depends on the position of the axes with respect to the curve. This is known

as ***transformation of coordinates.*** If the new axes are parallel to the old axes, respectively, the transformation is called ***transformation by translation.*** If the new axes have the same origin as the old axes but are oblique to them, the transformation is called ***transformation by rotation.***

If the coordinates of any point P are (x, y) when referred to one set of axes, and (x', y') when referred to the other set, then either set of coordinates can be expressed in terms of the other. Likewise, the equation of a given locus referred to one set of axes can be transformed into the equation of the same locus referred to the other set of axes.

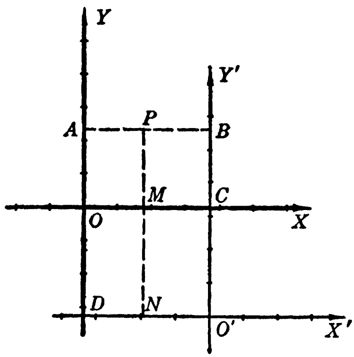


FIG. 34

21. Formulas of Translation. In Fig. 34 let the x and y axes be parallel to the x' and y' axes, respectively. Let the coordinates of O' be (h, k) when referred to the former set; that is, $OC = h$ and $CO' = k$. Let P be any point in the plane with coordinates (x, y) and (x', y'), respectively; that is,

$$AP = x, \qquad MP = y,$$
$$BP = x', \qquad NP = y'.$$

Now $AP = AB + BP$ and $MP = MN + NP$; hence

(X) $\quad \boldsymbol{x = x' + h, \qquad y = y' + k.}$

22. Formulas of Rotation. Let the x' axis make an angle θ with the x axis. Let P, any point in the plane, have coordinates (x, y) and (x', y'), respectively. Then $OM = x$, $MP = y$, $ON = x'$, and $NP = y'$. Now

$$OM = OT - SN,$$
$$MP = TN + SP.$$

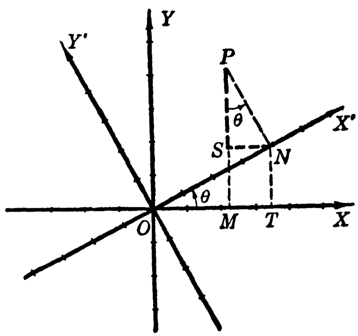


FIG. 35

But $OT = x' \cos \theta$; likewise $SN = y' \sin \theta$, $TN = x' \sin \theta$, and $SP = y' \cos \theta$. Hence

(XI) $\quad \boldsymbol{x = x' \cos \theta - y' \sin \theta, \qquad y = x' \sin \theta + y' \cos \theta.}$

EXAMPLES

1. Change the axes by translation so that the equation $2x^2 + 3x + 7 = y$ transforms into an equation without a term in x' and without a constant term.

SOLUTION. Translating the origin to (h, k), we have

$$2(x' + h)^2 + 3(x' + h) + 7 = y' + k.$$

Collecting terms, we find

$$2x'^2 + (4h + 3)x' + (2h^2 + 3h + 7 - k) = y'.$$

By the conditions of the problem we must have

$$4h + 3 = 0, \qquad 2h^2 + 3h + 7 - k = 0.$$

Hence $h = -3/4$, $k = +47/8$. That is, by translating the origin to $(-3/4, 47/8)$, the transformed equation is

$$2x'^2 = y'.$$

2. Find the transformed equation for $x^2 - y^2 = a^2$ when the axes are rotated through 45°.

SOLUTION. From formulas (XI) we have

$$x = \frac{1}{2}\sqrt{2}(x' - y'), \qquad y = \frac{1}{2}\sqrt{2}(x' + y').$$

Hence

$$\frac{1}{2}(x' - y')^2 - \frac{1}{2}(x' + y')^2 = a^2,$$

or

$$2x'y' + a^2 = 0.$$

3. By completing the squares, find the translation which will remove the first degree terms from the equation $2x^2 - 3y^2 - 4x - 2y = 13$.

SOLUTION. Completing the squares, we have

$$2(x - 1)^2 - 3\left(y + \frac{1}{3}\right)^2 = 13 + 2 - \frac{1}{3} = \frac{44}{3}.$$

The first degree terms will be eliminated if we make

$$x - 1 = x', \qquad y + \frac{1}{3} = y';$$

that is, if

$$x = x' + 1, \qquad y = y' - \frac{1}{3}.$$

Hence, if the origin is translated to the point $h = 1$, $k = -1/3$, the transformed equation will be

$$2x'^2 - 3y'^2 = \frac{44}{3}, \qquad \text{or} \qquad 6x'^2 - 9y'^2 = 44.$$

PROBLEMS

Simplify each of the following equations by translation. (Nos. 1–4.)

1. $x^2 - 6x + 4y = 10.$ *Ans.* $x'^2 + 4y' = 0.$
2. $3y = x^2 + 4x - 8.$
3. $x^2 + 3y^2 - 4x + 6y = 6$, to $(2, -1)$. *Ans.* $x'^2 + 3y'^2 = 13.$
4. $3x^2 - 2y^2 + 6x + 8y = 10$, to $(-1, 2)$.

Simplify each of the following equations by rotation. (Nos. 5–8.)

5. $xy = 12, \theta = -\pi/4.$ *Ans.* $y'^2 - x'^2 = 24.$
6. $x^2 - y^2 + 20 = 0, \theta = \pi/4.$
7. $2x^2 + xy\sqrt{3} + y^2 = 7, \theta = \pi/6.$ *Ans.* $5x'^2 + y'^2 = 14.$
8. $2x^2 - xy\sqrt{3} + y^2 = 10, \theta = \pi/3.$

Simplify each of the following equations by completing the squares. (Nos. 9–11.)

9. $x^2 + y^2 + x - y = 6.$ *Ans.* $2x'^2 + 2y'^2 = 13.$
10. $2x^2 + 3y^2 - 8x + 6y = 11.$
11. $6x^2 - 2y^2 - 9x + 8y = 10.$ *Ans.* $48x'^2 - 16y'^2 = 43.$

Apply translation and then rotation to simplify the equation. (No. 12.)

12. $x^2 + xy + y^2 - 2x + 2y = 2$, to $(2, -2)$ and then $\theta = \pi/4$.

ADDITIONAL PROBLEMS

Draw the graph of each of the following equations. These curves are important and many are of historic interest.

1. $r = a(1 + 2\cos\theta)$, *Trisectrix.*
2. $r = a \pm b \sin\theta$, *Limaçons* (if $a = b$, *cardioids*).
3. $r = a\cos 3\theta$, *Three-leafed rose.*
4. $r = a \sin 2\theta$, *Four-leafed rose.*
5. $x^3 + xy^2 = 2ay^2$, *Cissoid.*
6. $y = 8a^3/(x^2 + 4a^2)$, *Witch.*
7. $y^2/x^2 = (a + x)/(a - x)$, *Strophoid.*
8. $x = 3at/(1 + t^3), y = 3at^2/(1 + t^3)$, *Folium.*
9. $y = (a/2)(e^{x/a} + e^{-x/a})$, *Catenary.*
10. $y = a \sin kx + b \cos kx$, *Simple harmonic motion.*
11. $y = ae^{-b^2x^2}$, *Probability curve.*
12. $r = a\theta$, *Spiral of Archimedes.*
13. $\log r = a\theta$, *Logarithmic spiral.*

CHAPTER II

EQUATIONS OF DEFINED CURVES
EMPIRICAL EQUATIONS

23. The Straight Line. A straight line is determined by two fixed points P_1 and P_2 on it, or by one point P_1 and its slope m.

To find the equation of the line, we must obtain a relation connecting the given values x_1, y_1, m, and the coordinates of a point $P(x, y)$, which is assumed to be any point on the line.

Draw the line through P_1 with the given slope m. We know that the slope of a line may be obtained from the coordinates of the extremities of any segment of the line as in § 6. Hence, if the slope is m, we have the relation $m = (y - y_1)/(x - x_1)$, or

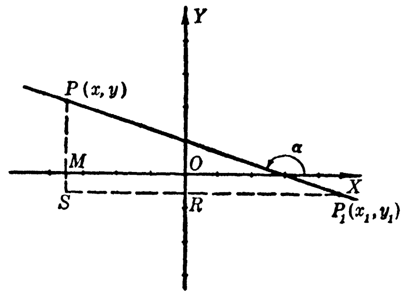


FIG. 36

(1) $$y - y_1 = m\,(x - x_1),$$

which is called the ***point-slope form*** of the equation of the line.

We may also derive equation (1) from Fig. 36 as follows:

$$\tan\alpha = m = \frac{SP}{P_1S} = \frac{SM + MP}{P_1R + RS},$$

or

$$m = \frac{-\,y_1 + y}{-\,x_1 + x}$$

for any position of $P(x, y)$ on the line.

Let the line be located by any two fixed points P_1 and P_2 and let $P(x, y)$ *be any other point on the line.* Since the slope of the segment P_1P equals the slope of the segment P_1P_2, we have

(2) $$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

which is called the ***two-point form*** of the equation of the line. This may also be derived from Fig. 37.

It is, however, preferable to use equation (1) even when two points are given. The right-hand member of (2) is m, therefore use this number m and either of the given points with the point $P(x, y)$ to write the point-slope form.

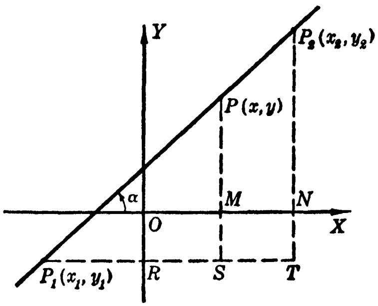


FIG. 37

Let the given point P_1 be the point where the line crosses the y axis. If the y intercept is b, then the point P_1 is $(0, b)$. Substituting these values for x_1 and y_1 in (1), we have

$$(3) \qquad y = mx + b,$$

called the ***slope-intercept form*** of the equation of the line.

If the line is given by its intersections with the axes, and if the x intercept is a and the y intercept b, these points are $(a, 0)$ and $(0, b)$ and the slope of the line is $m = OM/NO = b/(-a)$. Substituting this value of m in (3), we have

$$(4) \qquad \frac{x}{a} + \frac{y}{b} = 1,$$

which is known as the ***intercept form*** of the equation of the line.

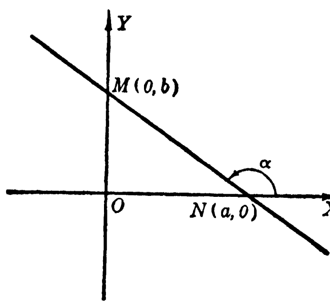


FIG. 38

It is important to realize that these are four forms of the same equation, and from the given data regarding the line we determine which form is the more convenient to use.

24. The Linear Equation. We shall now show that the general equation of the first degree in x and y, that is, the ***linear equation***

$$(1) \qquad Ax + By + C = 0,$$

where A, B, C are any constants, but A and B not both zero, always represents a straight line.

First, assume $B \neq 0$. Solving (1) for y, we have

$$(2) \qquad y = -\frac{A}{B}x - \frac{C}{B},$$

which is in the form of equation (3) of the preceding article, where $m = -A/B$ and $b = -C/B$. That is, *when* (1) *is solved for y the* ***coefficient*** *of x is then the* ***slope*** *of the line and the constant term is the y intercept.*

Second, assume $B = 0$. Then (1) reduces to $x = -C/A$, which is the equation of a line parallel to the y axis whose x intercept is $-C/A$. Since any line either crosses the y axis and has a slope and y intercept, or else is parallel to the y axis, it follows that every linear equation can be written in the form $y = mx + b$ or in the form $x = a$. Hence the equation $Ax + By + C = 0$ represents a straight line.

It is important to recognize the slope of a line from its equation. Since $m = -A/B$, we can say that *if $B \neq 0$, and if the terms in x and y are on the same side of the equality sign, the slope of the line is the quotient of the coefficient of x by the coefficient of y with its sign changed.*

EXAMPLES

1. Find the equation of the line through the point $(-3, -2)$ which makes an angle of 120° with the x axis.

SOLUTIONS. (a) The slope of the line is $m = \tan 120° = -\sqrt{3}$. If $P(x, y)$ is any point on the line, we have from the point-slope form of the equation of the line

$$-\sqrt{3} = \frac{y - (-2)}{x - (-3)},$$

or

$$x\sqrt{3} + y + 2 + 3\sqrt{3} = 0.$$

(b) Again, from Fig. 39 we have $\tan 120° = -\sqrt{3} = TP/P_1T$, or

$$-\sqrt{3} = \frac{TM + MP}{P_1N + NT},$$

$$-\sqrt{3} = \frac{2 + y}{3 + x},$$

$$x\sqrt{3} + y + 2 + 3\sqrt{3} = 0.$$

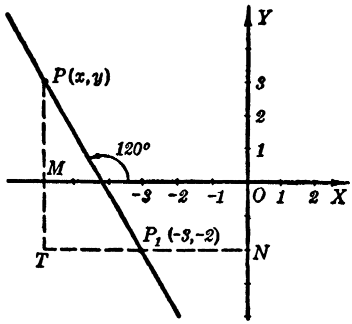


FIG. 39

2. Find the equation of the line through the points (2, 3) and $(-3, -1)$.

SOLUTIONS. (a) The slope of the line is $m = [3 - (-1)]/[2 - (-3)] = 4/5$. Then, from the point-slope form, if we use the point (2, 3),

$$\frac{4}{5} = \frac{y - 3}{x - 2}.$$

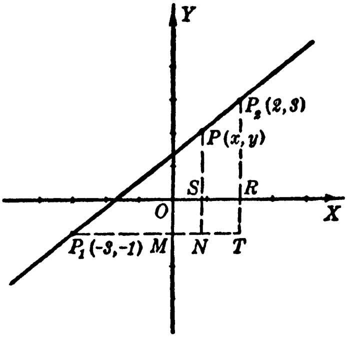


FIG. 40

Simplifying, we have

$$4x - 5y + 7 = 0.$$

(b) By line-segments, from Fig. 40,

$$\frac{TP_2}{P_1T} = \frac{NP}{P_1N},$$

$$\frac{TR + RP_2}{P_1M + MT} = \frac{NS + SP}{P_1M + MN},$$

$$\frac{1+3}{3+2} = \frac{1+y}{3+x},$$

or

$$4x - 5y + 7 = 0.$$

3. Find the equation of the line crossing the y axis 4 units above the origin and making with the y axis an angle of 60°.

SOLUTION. From Fig. 41, we see that the inclination of the line is 150° or 30. Hence $m = \tan 150° = -\sqrt{3}/3$, or $m = \tan 30° = \sqrt{3}/3$. From either the slope-intercept form or the sides of the triangles, we have the equations of the lines as

$$x \pm y\sqrt{3} \mp 4\sqrt{3} = 0.$$

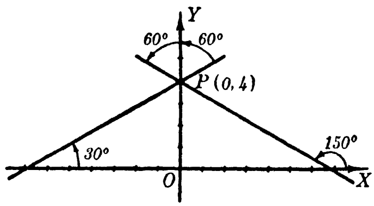


FIG. 41

4. Find the equation of the line through $(-2, 3)$ which is perpendicular to the line $2x - 3y - 3 = 0$.

SOLUTION. The slope of the given line is $m = -[2/(-3)] = 2/3$. Hence the slope of the required line is $-3/2$. The equation of the line by either method outlined in Example 1 is

$$3x + 2y = 0.$$

25. Parallel and Perpendicular Lines. Given the two lines, L_1 with equation $A_1x + B_1y + C_1 = 0$, and L_2 with equation $A_2x + B_2y + C_2 = 0$. Call the slopes of these two lines m_1 and m_2, respectively. It follows then that $m_1 = -A_1/B_1$ and $m_2 = -A_2/B_2$, where B_1 and B_2 are both different from zero.

The condition that L_1 and L_2 be parallel is (§ 7) that the slopes be equal. Hence $m_1 = m_2$, that is, $-A_1/B_1 = -A_2/B_2$, or

$$\frac{A_1}{A_2} = \frac{B_1}{B_2}. \tag{1}$$

Therefore, *if two lines are parallel the coefficients of x and y in the equations of the lines are proportional, and conversely.*

If L_1 and L_2 are perpendicular, $m_1m_2 = -1$ (§ 7). Therefore $(-A_1/B_1)(-A_2/B_2) = -1$, or

$$A_1A_2 + B_1B_2 = 0. \tag{2}$$

Conversely, if this relation exists, the lines L_1 and L_2 are perpendicular.

26. Special Cases. If, in the equation $Ax + By + C = 0$, we have $A = 0$, the equation becomes $y = -C/B$, which is in the form $y = b$ and is a line parallel to the x axis. If $B = 0$, the line is $x = -C/A$, parallel to the y axis. If $C = 0$, the line is $Ax + By = 0$ or $y = (-A/B)x$. This equation is satisfied by the point $(0, 0)$; hence the line passes through the origin. The converse is also true; that is, if the line passes through the origin, the constant term in its equation is zero.

PROBLEMS

Find the equation of each of the following lines. (Nos. 1–15.)

1. Through $(-3, 4)$ and $(4, -5)$. *Ans.* $9x + 7y = 1$.

2. Through $(2, -4)$ with slope $-6/5$.

3. With x intercept -4 and inclination of $\pi/4$. *Ans.* $x - y + 4 = 0$.

4. With y intercept -3 and slope $2/3$.

5. Through $(-2, -4)$ and parallel to the line joining the origin to $(3, 2)$. *Ans.* $2x - 3y = 8$.

6. Through $(-3, 5)$ and perpendicular to line joining $(-3, 5)$ and $(-5, 4)$.

7. With x and y intercepts $-3/7$ and $5/2$, respectively. *Ans.* $35x - 6y + 15 = 0$.

8. Through $(5, -3)$ with $\alpha = 135°$.

9. Through $(-4, 3)$ with $\alpha = 120°$. *Ans.* $\sqrt{3}\,x + y + 4\sqrt{3} - 3 = 0$.

10. Through $(6, -1)$ with an inclination of $150°$.

11. Perpendicular to the line $3x - 4y = 5$ and through $(-5, 6)$. *Ans.* $4x + 3y + 2 = 0$.

12. Parallel to the line $7x + 6y = 12$ and through $(-7, 3)$.

13. Through $(-2, -3)$ and making an angle of $45°$ with the line $y = 7x + 11$. *Ans.* $3x - 4y = 6$, and $4x + 3y + 17 = 0$.

14. Through $(5, -3)$ and the intersection of the lines $3x - 4y = 7$ and $2x + y = 1$.

15. Through $(-3, 4)$ and the mid-point of the segment joining $(4, 2)$ and $(-1, -7)$. *Ans.* $13x + 9y + 3 = 0$.

16. Find the equation of the perpendicular bisector of the segment joining $(-2, 1)$ and $(-6, -5)$.

17. A diagonal of a square has its extremities at $(-1, -1)$ and $(3, 5)$; find the equations of its sides.

Ans. $x - 5y = 4,\ 5x + y + 6 = 0,\ x - 5y + 22 = 0,\ 5x + y = 20.$

18. The vertex of the right angle of an isosceles right triangle is at $(3, -1)$ and another vertex is at $(5, 4)$. Find the equations of the three sides.

19. The sides of a square have equations $x + 3y = 10$, $x + 3y = 20$, $3x - y + 5 = 0$, and $3x - y = 5$. Find the equations of the diagonals and find their point of intersection.

Ans. $\begin{cases} 2x - 4y + 15 = 0, \\ 4x + 2y = 15, \end{cases}$ $(3/2, 9/2)$.

20. The vertices of a triangle are at $(3, 3)$, $(-1, -5)$, and $(6, 0)$. Find the equation of the line through each vertex parallel to the opposite side.

21. Find the perpendicular distance between the parallel lines $3x = y - 2$ and $6x - 2y - 1 = 0$. *Ans.* $\sqrt{10}/4$ units.

22. Find the length of the perpendicular from the intersection of the two lines $x - 6y = 1$ and $10y - x + 5 = 0$ to the line $x + 2y + 1 = 0$.

23. Find the equations of the locus of points equidistant from the lines $x = a$ and $y = b$. *Ans.* $x - y = a - b,\ x + y = a + b.$

24. Given the triangle $A(-4, -2)$, $B(6, 4)$, $C(-2, 8)$.

(*a*) Find the equations of its sides.
(*b*) Find the equations of the medians.
(*c*) Find the equations of the altitudes.
(*d*) Find the equations of the perpendicular bisectors of the sides.
(*e*) Find the intersection of the medians (*centroid*).
(*f*) Find the intersection of the altitudes (*orthocenter*).
(*g*) Find the intersection of the perpendicular bisectors (*circumcenter*).
(*h*) Prove that the points (*e*), (*f*), and (*g*) are collinear (*Euler line*).

27. The Circle. *A circle is the locus of a point at a given distance from a fixed point.*

Let (h, k) be the fixed point called the ***center,*** and let r be the given distance which is the ***radius.*** From the definition any point $P(x, y)$ on the locus must satisfy the relation $CP = r$, where C is the center (h, k). From the right triangle in Fig. 42, we have

$$\begin{aligned} r^2 &= \overline{CP}^2 = \overline{CR}^2 + \overline{RP}^2 \\ &= (CN + NR)^2 + (RM + MP)^2 \\ &= (-h + x)^2 + (-k + y)^2. \end{aligned}$$

This relation, which may be obtained from the distance formula, gives the equation of the circle as

$$(1) \qquad (x - h)^2 + (y - k)^2 = r^2.$$

If we expand this equation and collect terms, we have

$$x^2 + y^2 - 2\,hx - 2\,ky + h^2 + k^2 - r^2 = 0,$$

which is of the form

$$(2) \qquad x^2 + y^2 + Dx + Ey + F = 0.$$

These are the two standard forms of the equation of a circle, and since (h, k) may be any point, and r any distance, this equation may represent any circle in the plane. Since any circle can be written in either of these forms, it follows that *an equation of the second degree represents a circle only when the coefficients of x^2 and y^2 are the same and the equation has no term in xy.*

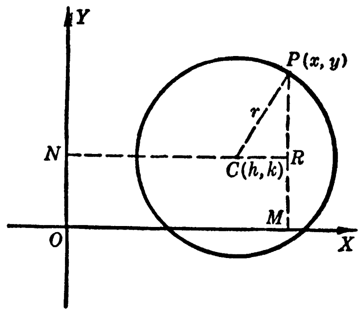


FIG. 42

We note the following special positions of the circle with respect to the coordinate axes:

If the equation has no term in x, then $D = -2\,h = 0$, or $h = 0$, which means that the center lies on the y axis. Similarly, if there is no term in y, then $k = 0$ and the center is on the x axis. If the terms in x and y are both missing, the equation reduces to

$$(3) \qquad x^2 + y^2 = r^2,$$

and the center is the origin. Conversely, if the center is at the origin the equation of the circle will be in form (3).

If $F = 0$, then $h^2 + k^2 = r^2$, and the circle will pass through the origin.

If the equation is given in form (2), it can be reduced to form (1) by completing the squares in x and y.

Since equations (1) and (2) each have three constants, we see that three conditions, which may be expressed as algebraic relations connecting these constants, are necessary and sufficient to determine the equation of the circle.

EXAMPLES

1. Find the center and radius of the circle whose equation is

$$2\,x^2 + 2\,y^2 - 10\,x + 6\,y = 23.$$

SOLUTION. Divide each member of the equation by 2 and complete the squares. Then

$$\left(x^2 - 5x + \frac{25}{4}\right) + \left(y^2 + 3y + \frac{9}{4}\right) = \frac{23}{2} + \frac{25}{4} + \frac{9}{4},$$

or

$$\left(x - \frac{5}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = 20.$$

Hence the center is $(5/2, -3/2)$ and the radius is $\sqrt{20}$ units.

2. Find the equation of the circle through the points $(0, 0)$, $(-1, 3)$, and $(5, -3)$.

SOLUTION. Since nothing is known of the center or radius, we shall use the form (**2**) of the equation of the circle. Substituting the coordinates of each point in that equation successively, we have

$$\begin{cases} F = 0, \\ 10 - D + 3E + F = 0, \\ 34 + 5D - 3E + F = 0. \end{cases}$$

Solving these, we find that $D = -11$ and $E = -7$. Hence the desired equation is

$$x^2 + y^2 - 11x - 7y = 0.$$

3. Find the equation of the circle through the point $(-6, 2)$ which is tangent to the y axis and has its center on the line $x + y = -1$.

SOLUTION. Since the circle passes through $(-6, 2)$, it must lie to the left of the y axis; hence $h = -r$. Then its equation has the form

$$(x + r)^2 + (y - k)^2 = r^2.$$

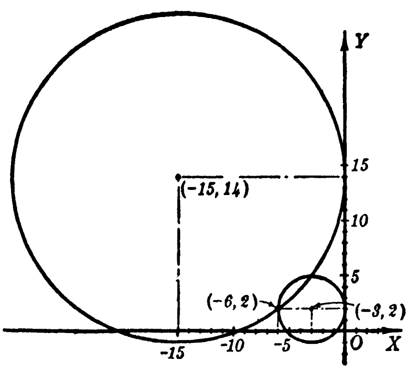


FIG. 43

Using the point $(-6, 2)$, we may write

$$(-6 + r)^2 + (2 - k)^2 = r^2.$$

Also, since the center is on the given line, we have

$$h + k + 1 = 0,$$

or

$$-r + k + 1 = 0.$$

From these relations we have

$$k^2 - 16k + 28 = 0,$$

whence

$$k = 2, 14; \qquad h = -3, -15; \qquad r = 3, 15.$$

Hence there are two circles satisfying the given conditions; and they are

$$(x + 3)^2 + (y - 2)^2 = 9, \quad \text{and} \quad (x + 15)^2 + (y - 14)^2 = 225.$$

4. Prove that an angle inscribed in a semicircle is a right angle.

Solution. Take the origin at the center and the x axis along the given diameter LM. Let $A(a, b)$ be any point on the circumference with m_1 and m_2 the slopes of AL and AM respectively. Then $m_1 = b/(a + r)$, $m_2 = b/(a - r)$; whence we have $m_1m_2 = b^2/(a^2 - r^2)$. But we must use the fact that A is on the circle, which gives $a^2 + b^2 = r^2$. Substituting $-b^2$ for $a^2 - r^2$ we get $m_1m_2 = -1$. Hence AL and AM are perpendicular (§ 7).

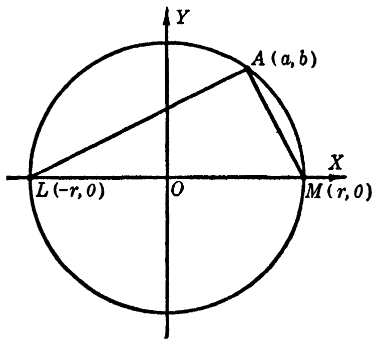


Fig. 44

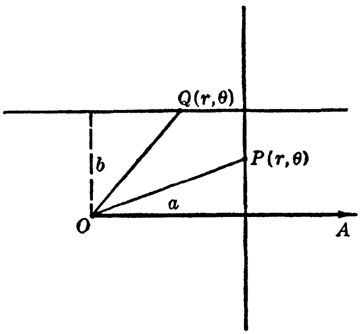


Fig. 45

28. The Line and the Circle in Polar Coordinates. The equation of any line perpendicular to the polar axis and at a distance a from the pole is

$$(1) \qquad r \cos \theta = a.$$

If the line is parallel to the polar axis and at a distance b from it, the equation is

$$(2) \qquad r \sin \theta = b.$$

These equations are the transformations from rectangular to polar coordinates of $x = a$ and $y = b$ [§ 18, (VIII)], where the pole is the origin and the polar axis is the x axis.

If the center of a circle of radius a is at the pole, its equation is obviously

$$(3) \qquad r = a.$$

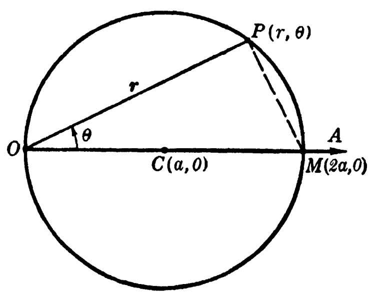


Fig. 46

If the center is on the polar axis with the pole at one extremity of a diameter, then any point $P(r, \theta)$ on the circle will satisfy the relation

$$(4) \qquad r = 2\, a \cos \theta.$$

Similarly, if the center is on the 90° vector with the pole on the circle we have

(5) $$r = 2a \sin \theta.$$

If the pole is on the circle and the intercepts on the polar axis and 90° vector are c and d respectively, the polar equation is readily found by transforming the corresponding rectangular equation $x^2 + y^2 - cx - dy = 0$. Thus (Fig. 47)

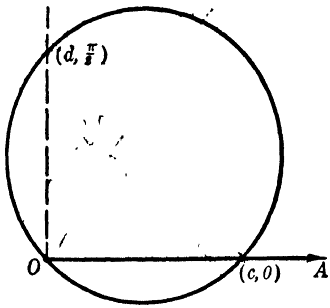


FIG. 47

(6) $$r = c \cos \theta + d \sin \theta.$$

29. The Line and the Circle in Parametric Form. If x and y are each given in terms of a third variable t and if each is of the first degree in the variable, the rectangular relation connecting x and y is linear. Thus if $x = at + b$ and $y = ct + d$, solving each for t and equating the results, we have

$$cx - ay = bc - ad.$$

Each value of t assigns one value to x and one to y. These values are the coordinates of a point on the line corresponding to t.

Given a circle whose center is the origin, draw the radius to any point $P(x, y)$ on the circumference. Denoting by θ the angle which the radius OP makes with the x axis, we have (Fig. 48)

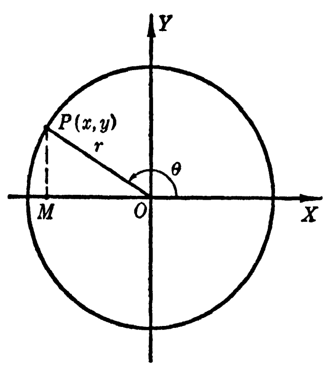


FIG. 48

(1) $$x = r \cos \theta, \quad y = r \sin \theta,$$

as ***parametric equations of the circle.*** The elimination of θ, as explained in § 19, gives the equation

$$x^2 + y^2 = r^2.$$

PROBLEMS

Find the center and radius of each of the following circles. (Nos. 1–6.)

1. $x^2 + y^2 - 10\,y = 0.$ *Ans.* (0, 5), 5 units.

2. $x^2 + y^2 + 6\,x - 4\,y = 7.$

3. $x^2 + y^2 - 16\,x + 12\,y + 1 = 0.$ *Ans.* $(8, -6)$, $3\sqrt{11}$ units.

4. $x^2 + y^2 - 5\,x + 4\,y = 2.$

5. $4\,x^2 + 4\,y^2 - 16\,x + 24\,y + 51 = 0.$ *Ans.* $(2, -3)$, 1/2 unit.

6. $36\,x^2 + 36\,y^2 - 180\,x + 48\,y + 97 = 0.$

Find the polar equation of the circle in the following two problems. (Nos. 7–8.)

7. With center at $(2, \pi/2)$ and radius 2 units. *Ans.* $r = 4 \sin \theta.$

8. Passing through the origin with center at (4, 0).

Find the equation of the circle satisfying each of the following sets of conditions and draw its graph. (Nos. 9–16.)

9. With center at $(-3, 4)$ and passing through (1, 1).
Ans. $x^2 + y^2 + 6\,x - 8\,y = 0.$

10. With center at $(-2, 4)$ and passing through the origin.

11. With center on the line $2\,x - 3\,y + 2 = 0$ and passing through the points $(4, -2)$ and (11, 3). *Ans.* $x^2 + y^2 - 10\,x - 8\,y + 4 = 0.$

12. Tangent to the x axis, with one extremity of a horizontal diameter at the point $(2, -2)$. (Two cases.)

13. Concentric with the circle $x^2 + y^2 - 5x + 4y = 1$, and passing through $(-2, 4)$. *Ans.* $x^2 + y^2 - 5\,x + 4\,y = 46.$

14. Through the points (0, 3), $(-3, 0)$ and (0, 0).

15. Through the points (1, 0), (2, 0) and (0, 3).
Ans. $3\,x^2 + 3\,y^2 - 9\,x - 11\,y + 6 = 0.$

16. Which has the segment of $2\,x - 3\,y + 6 = 0$ cut off by the reference lines as a diameter.

17. Find the locus of the vertex of a right angle if its sides pass through the points (1, 1) and (1, 7), respectively. *Ans.* $x^2 + y^2 - 2\,x - 8\,y + 8 = 0.$

18. Find the locus of points four times as far from $(-2, 1)$ as from $(1, -3)$.

19. Transform into polar coordinates the equation of the circle through the origin with center $(-4, 2)$. *Ans.* $r + 8 \cos \theta - 4 \sin \theta = 0.$

20. Find the locus of a point if the square of its distance from the point $(1, -3)$ is six times its distance from the y axis.

21. Find the points at which the y axis cuts the circle having the segment from (2, 1) to $(-4, -3)$ as a diameter. *Ans.* $(0, -1 \pm 2\sqrt{3})$.

22. Find the circles passing through $(-3, 4)$ and tangent to both axes.

23. Find the circle with center at $(-5, 1)$ and tangent to the line $4x + 3y = 3$. *Ans.* $x^2 + y^2 + 10x - 2y + 10 = 0$.

24. Find the circle through the origin and tangent to the line $5x - 2y = 16$ at the point $(2, -3)$.

25. Chords of the circle $r + 10 \cos \theta = 0$ are drawn through the pole. Find the locus of their mid-points. *Ans.* $r + 5 \cos \theta = 0$.

26. Chords of the circle $r = 6 \sin \theta$ are drawn through the pole and each produced its own length. Find the locus of the extremities.

27. A chord of the circle $x^2 + y^2 + 2x + 4y = 44$ has its mid-point at $(-3, 1)$. Find the equation of the chord and its length.
Ans. $2x - 3y + 9 = 0$; 12 units.

28. The equations of two circles are $x^2 + y^2 + D_1x + E_1y + F_1 = 0$ and $x^2 + y^2 + D_2x + E_2y + F_2 = 0$. Find the relation connecting these coefficients if the circles are (*a*) equal; (*b*) concentric; (*c*) tangent to each other.

30. The Parabola. *A* ***parabola*** *is the locus of points equidistant from a fixed point and a fixed line.* The fixed point is called the ***focus*** and the fixed line the ***directrix.***

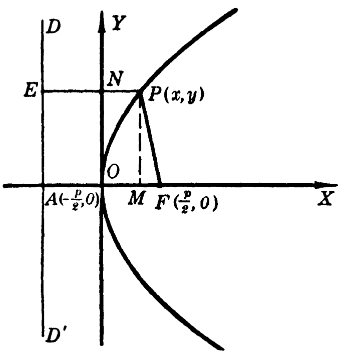


FIG. 49

Let F be the focus and $D'D$ the directrix, with p the distance between the focus and the directrix. Through F draw AF perpendicular to the directrix.

The x and y axes may be taken in any convenient position with respect to the focus and directrix, and the corresponding equation of the curve obtained. However, the simplest and standard form of the equation is found by taking the origin midway between the focus and directrix with the line AF as one of the axes, say the x axis. If $P(x, y)$ is any point on the parabola, from the definition we have $EP = FP$. But

$$EP = EN + NP = \frac{p}{2} + x,$$

and

$$FP = \sqrt{\overline{MF}^2 + \overline{MP}^2} = \sqrt{(OF - OM)^2 + \overline{MP}^2}$$
$$= \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2}.$$

Hence

$$\sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = \frac{p}{2} + x.$$

Squaring and collecting terms, we get the form

$$(1) \qquad y^2 = 2\,px.$$

Since we see that the curve is symmetric to the x axis, the line AF is called the ***axis of the parabola.*** The intersection of the curve and its axis is the ***vertex.*** The curve is open and extends indefinitely to the right of the y axis.

Any chord of the parabola through F is called a ***focal chord.*** The focal chord perpendicular to the axis of the parabola is called the ***latus rectum;*** its length is $2\,p$.

Equation (1) may be written in the form

$$(2) \qquad x = ky^2,$$

where k is any constant not zero. The graph extends to the right or to the left from the origin according as k is positive or negative.

In a similar manner the equation

$$(3) \qquad y = kx^2,$$

where k is any constant not zero, is a parabola with vertex at the origin and with its axis along the y axis. It is concave upward or concave downward according as k is positive or negative.

31. Other Forms of the Equation of the Parabola. Two important equations of the parabola are

$$(1) \qquad y = ax^2 + bx + c,$$

and

$$(2) \qquad x = ay^2 + by + c.$$

The former equation defines a parabola with its axis parallel to the y axis. For, on completing the square, we have

$$y = a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4\,a^2}\right] + c - \frac{b^2}{4\,a},$$

or

$$y - \left(c - \frac{b^2}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2.$$

From this equation, we see that a translation of the origin to the point $(-b/2a, c - b^2/4a)$ will reduce equation (1) to the form (see § 21)

$$y' = ax'^2. \tag{3}$$

Likewise equation (2) represents a parabola with axis parallel to the x axis.

If we substitute for x and y in either equation (1) or (2) the coordinates of any three distinct points not on the same straight line, we obtain three corresponding relations connecting the coefficients a, b, and c. The solution of these relations will determine a unique set of values for a, b, and c. Hence:

Through any three distinct points not on the same straight line, one and only one parabola can be drawn with a vertical axis, and one and only one parabola can be drawn with a horizontal axis.

32. Construction of the Parabola. When the focus and directrix are given, points on the parabola may be found readily by means of a ruler and compasses. Draw any line parallel to the directrix and on the same side as the focus. Let its distance from the directrix be a. With F as a center and a radius a, mark the points on the given line at a distance a from F. These are points on the parabola. If the directrix is taken parallel to either set of ruled lines of the coordinate paper, points on the curve can be marked rapidly with the compasses, and an accurate sketch of the curve can be drawn.

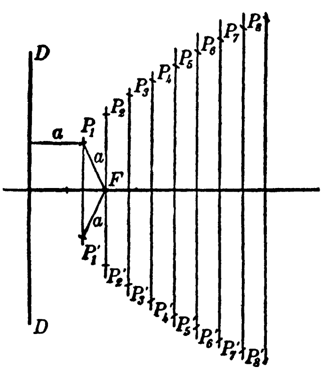


FIG. 50

33. Parabolic Segment. A segment cut from a parabola by a chord perpendicular to the axis is known as a ***parabolic segment.*** If we take the curve $y = kx^2$ and draw through two points P_1 and P_2 the chords perpendicular to the axis (Fig. 51), we have

$$y_1 = kx_1^2, \qquad y_2 = kx_2^2;$$

whence

$$\frac{y_1}{y_2} = \frac{x_1^2}{x_2^2}, \quad \text{or} \quad \frac{y_1}{y_2} = \frac{(2\,x_1)^2}{(2\,x_2)^2}.$$

But $2\,x_1$ and $2\,x_2$ are the lengths of the two chords and y_1 and y_2 are their respective distances from the vertex. Hence without reference to the coordinate axes we have the theorem:

In any parabola the squares of any two chords which are perpendicular to its axis are to each other as their distances from the vertex of the parabola. In the segment Q_1OP_1, Q_1P_1 is called the ***base*** and ON_1 the ***altitude*** of the segment. In practical problems the base of the parabolic arch is usually called the ***span.***

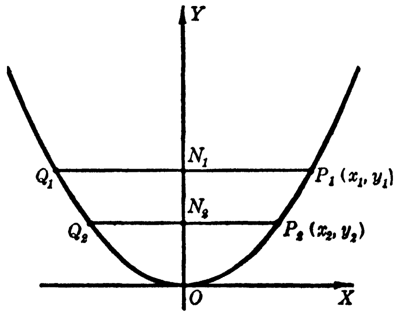


FIG. 51

EXAMPLES

1. Find the equation of the parabola with $y = 6$ as directrix and $(-4, 2)$ as focus. Find the vertex and extremities of the latus rectum.

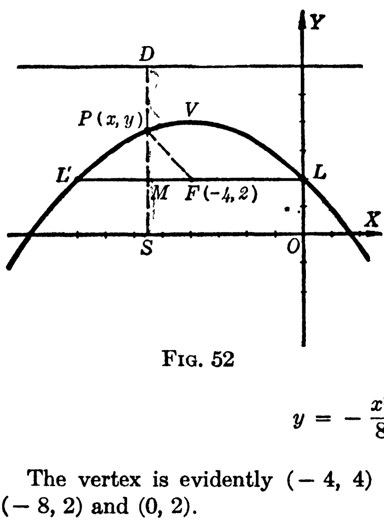


FIG. 52

SOLUTION. Assume $P(x, y)$ any point on the parabola. Draw PF and draw PD perpendicular to the directrix. From the definition, we have $PD = FP$. But

$$PD = PS + SD = -y + 6,$$

and

$$FP = \sqrt{(FL + LM)^2 + (MS + SP)^2}$$
$$= \sqrt{(4 + x)^2 + (-2 + y)^2}.$$

Hence

$$(x + 4)^2 + (y - 2)^2 = (6 - y)^2,$$

or

$$y = -\frac{x^2}{8} - x + 2.$$

The vertex is evidently $(-4, 4)$ and the ends of the latus rectum are $(-8, 2)$ and $(0, 2)$.

2. In the parabolic segment shown in Fig. 53, find x.

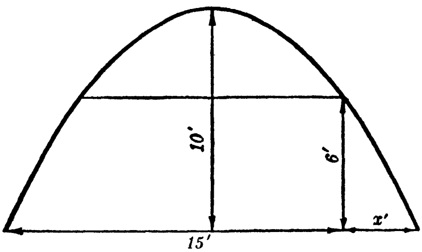


Fig. 53

Solution. It is evident that the two chords are, respectively, $15 + x$ and $15 - x$ feet. Hence

$$\frac{(15 + x)^2}{(15 - x)^2} = \frac{10}{4}, \quad \text{or} \quad \frac{15 + x}{15 - x} = \frac{\sqrt{10}}{2}.$$

Solving for x, we find

$$x = 5(7 - 2\sqrt{10}) = 3.38 \text{ feet.}$$

PROBLEMS

Find the focus, directrix, and extremities of the latus rectum of the following parabolas. (Nos. 1–8.)

1. $x^2 - 2y = 0$. *Ans.* $(0, 1/2)$, $2y + 1 = 0$, $(\pm 1, 1/2)$.

2. $x^2 + 12y = 0$.

3. $y^2 + 9x = 0$. *Ans.* $(-9/4, 0)$, $4x = 9$, $(-9/4, \pm 9/2)$.

4. $y^2 - 6x = 0$.

5. $4x^2 + 9y = 0$. *Ans.* $(0, -9/16)$, $16y = 9$, $(\pm 9/8, -9/16)$.

6. $5y^2 + 16x = 0$.

7. $x^2 + 2py = 0$.

8. $y^2 = kx$.

Derive the equation of the parabola which has the following given parts. (Nos. 9–16.)

9. Vertex at the origin, focus at $(0, -5/2)$. *Ans.* $x^2 + 10y = 0$.

10. Focus at $(-9/4, 0)$, directrix $x = 9/4$.

11. Vertex at $(5, 1)$ and directrix $y = -3$.
Ans. $x^2 - 10x - 16y + 41 = 0$.

12. Vertex at $(2, -1)$ and focus at $(-1, -1)$.

13. Focus on the x axis, vertex at the origin, passes through $(4, -3)$.
Ans. $4y^2 = 9x$.

14. Vertex at the origin, directrix parallel to the x axis, passes through the point $(3, -1)$.

15. Focus at $(-2, 4)$, one extremity of the latus rectum at $(3, 4)$.
Ans. $x^2 + 4x + 10y = 61$, or $x^2 + 4x - 10y + 19 = 0$.

16. Extremities of the latus rectum at $(1, 6)$ and $(1, -4)$. (*Two cases.*)

17. Find the locus of a point whose distance from $(2, 3)$ is two units more than its distance from the line $x + 5 = 0$. *Ans.* $y^2 - 6y - 18x = 36$.

18. Find the locus of a point whose distance from the point $(-2, 8)$ is three units less than its distance from the x axis.

19. Find x in the parabolic segment, Fig. 54. *Ans.* 2.252 ft.

20. Find AC in the parabolic segment, Fig. 55.

21. Find h in the parabolic segment, Fig. 56. *Ans.* 10.8 ft.

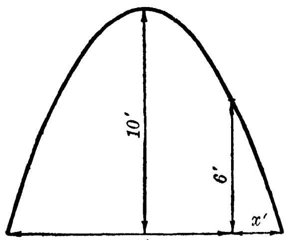


FIG. 54

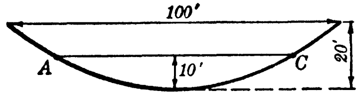


FIG. 55

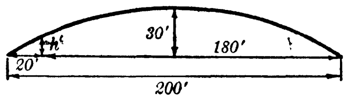


FIG. 56

22. A parabolic arch of height h spans a stream of width w. (*a*) What part of the width of the stream has a clearance of at least $(3/4)\,h$? (*b*) What part has a clearance less than $(h/2)$? (*c*) What do these results become if $h = 24$ feet and $w = 150$ feet?

23. Find the height of a parabolic segment if $AO = OB = 60'$, $CB = 10'$, and $CD = 12'$. (See Fig. 57). *Ans.* $39\frac{3}{11}$ ft.

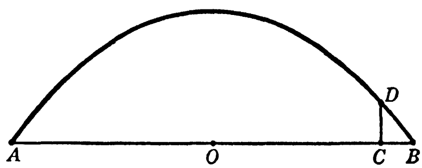


FIG. 57

24. Find the height of a parabolic arch if its base is 24 feet and if there is a clearance of 10 feet at a distance of 4 feet from the end of the base line.

25. Find the equation of the parabola with a vertical axis which passes through the points $(-5, 1)$, $(3, -1)$, and $(0, -4)$.
Ans. $4y = x^2 + x - 16$.

26. Find the equation of the parabola with a horizontal axis which passes through the three points in Problem 25.

27. Find the equation of the parabola with a horizontal axis which passes through the points $(-3, 0)$, $(1, 4)$, and $(6, -4)$.
Ans. $32x = 13y^2 - 20y - 96$.

28. Find the equation of the parabola with a vertical axis which passes through the three points in Problem 27.

Find the vertex of each of the following parabolas. Translate the origin to the vertex and find the transformed equation.

29. $y^2 - 8x + 6y + 17 = 0$. *Ans.* $(1, -3)$; $y'^2 = 8x'$.

30. $2x = 8 + 4y - y^2$.

31. $y = 6 - 2x - x^2/3$. *Ans.* $(-3, 9)$; $x'^2 + 3y' = 0$.

32. $y = 2x^2 - 5x - 3$.

34. The Ellipse. *An* ***ellipse*** *is the locus of a point the sum of whose distances from two fixed points is a constant.*

Let F' and F be the two fixed points at a distance of $2c$ from each other. These points are called the ***foci.*** Let the sum of the distances of any point $P(x, y)$ on the ellipse from F' and F be $2a$. Take O, the mid-point of $F'F$, as the origin and $F'F$ along the x axis. The coordinates of the foci are then $(\pm c, 0)$. Any point P on the ellipse must satisfy the definition

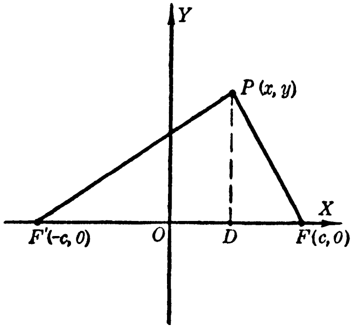


FIG. 58

$$F'P + FP = 2a.$$

Then, using the distance formula, or from Fig. 58, we have

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

Transposing one radical, say the second, squaring both sides, and collecting terms, we find

$$4cx = 4a^2 - 4a\sqrt{(x-c)^2 + y^2}.$$

Solving for the radical, we have

$$\sqrt{(x-c)^2 + y^2} = a - \frac{cx}{a}.$$

Squaring again, we obtain, after collecting terms,

$$\frac{(a^2 - c^2)x^2}{a^2} + y^2 = a^2 - c^2,$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

From Fig. 58 it is obvious that $2a > 2c$ or $a > c$; then if we set $a^2 - c^2 = b^2$ it follows that b^2 is positive and hence b is real for every ellipse. The equation takes the form

$$\text{(1)} \qquad \boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.}$$

This equation shows that the ellipse is symmetric with respect

to both axes and to the origin. The x intercepts are $\pm a$, the y intercepts $\pm b$. Solving for y and x in turn, we get

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2},$$

$$x = \pm \frac{a}{b}\sqrt{b^2 - y^2}.$$

These forms show that no real point of the curve can have an abscissa numerically greater than a, nor an ordinate numerically greater than b. The ellipse is a closed curve and is inscribed in the rectangle whose sides are $x = \pm a$, $y = \pm b$. The segment $V'V$ is called the ***major axis,*** its extremities V' and V being the ***vertices.*** The segment $B'B$ is the ***minor axis.*** The length of the major axis is $2a$, and the length of the minor axis is $2b$.

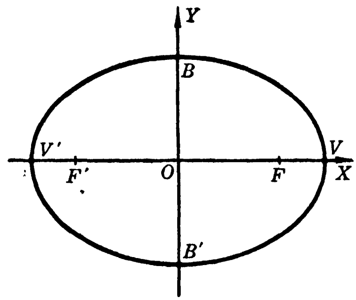


FIG. 59

The chord through either focus perpendicular to the major axis is called a ***latus rectum.*** To find its length, set $x = \pm c$ in equation (1), then $y = \pm b^2/a$. Hence the length of the latus rectum is $2b^2/a$.

If the foci are on the y axis, the equation of the ellipse is

$$(2) \qquad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

35. Limiting Forms. Eccentricity. If $c = 0$, since we have $a^2 - c^2 = b^2$, it follows that $a = b$. That is, if F' and F are made to coincide at the center, the major and minor axes become equal, and the equation becomes

$$x^2 + y^2 = a^2.$$

Hence the circle is the limiting form of the ellipse as the foci approach coincidence.

On the other hand, if F' and F approach V' and V, respectively, when they coincide $c = a$ and $b = 0$. Hence, as c approaches a and b approaches 0, the ellipse flattens and approaches as a limiting form the line-segment $V'V$.

The shape of the ellipse depends on the relative lengths of

c and a. The ratio c/a is called the ***eccentricity*** of the ellipse. It is denoted by e and may have any value between 0 and 1, these two extremes being the eccentricities for the limiting forms of the circle and the line-segment, respectively.

36. More General Forms of the Equation of an Ellipse. It is important to recognize that the equation

$$(1) \qquad Ax^2 + By^2 = C$$

represents an ellipse provided A, B, and C have the same sign. Writing this in the form

$$\frac{x^2}{C/A} + \frac{y^2}{C/B} = 1,$$

we see that the major axis is along the x axis or the y axis according as $A < B$ or $A > B$.

Completing the squares, we may write the equation

$$(2) \qquad Ax^2 + By^2 + Dx + Ey + F = 0,$$

where A and B have like signs, in the form

$$A\left(x + \frac{D}{2A}\right)^2 + B\left(y + \frac{E}{2B}\right)^2 = C',$$

where $C' = D^2/(4A) + E^2/(4B) - F$. Hence, translating the origin to the center $(-D/2A, -E/2B)$, we obtain the transformed equation

$$Ax'^2 + By'^2 = C',$$

which is like equation (1) if C' has the same sign as A and B.

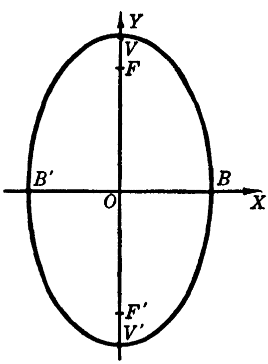


FIG. 60

EXAMPLES

1. Given the ellipse $25x^2 + 9y^2 = 196$. Find the coordinates of the vertices and foci, and find the eccentricity.

SOLUTION. Dividing both sides by 196, we have

$$\frac{x^2}{\left(\frac{14}{5}\right)^2} + \frac{y^2}{\left(\frac{14}{3}\right)^2} = 1.$$

Hence the major axis is along the y axis, and $a = 14/3$, $b = 14/5$. Then $c = 56/15$. The vertices are $(0, \pm 14/3)$; the foci are $(0, \pm 56/15)$; and $e = c/a = 4/5$.

2. Derive the equation of the ellipse if its foci are at $(-2, 3)$ and $(6, 3)$ with one vertex at $(-4, 3)$.

SOLUTION. The center of the ellipse is the mid-point between the foci or (2, 3). Hence the semi-major axis a is 6, so the other vertex is (8, 3).

If $a = 6$, and $c = 4$, then $b = 2\sqrt{5}$. The extremities of the minor axis are $(2, 3 + 2\sqrt{5})$ and $(2, 3 - 2\sqrt{5})$. By the definition of the ellipse, we see that the sum of the distances from any point $P(x, y)$ on the curve to $(-2, 3)$ and $(6, 3)$ is $2a = 12$. Hence we find $F'P + FP = 12$. That is,

$$\sqrt{(x+2)^2 + (y-3)^2} + \sqrt{(x-6)^2 + (y-3)^2} = 12.$$

Rationalizing, we have

$$5x^2 + 9y^2 - 20x - 54y - 79 = 0.$$

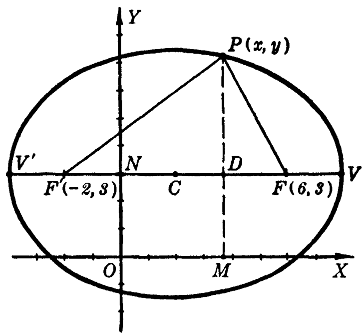


FIG. 61

3. Change the equation of Example 2 to the form of (1) by completing squares.

SOLUTION. Write the equation in the form

$$5(x-2)^2 + 9(y-3)^2 = 79 + 20 + 81 = 180.$$

Then a translation of the origin to the center (2, 3) gives

$$5x'^2 + 9y'^2 = 180.$$

PROBLEMS

Find the coordinates of the vertices and foci, the eccentricity, the lengths of the major and minor axes, and sketch each of the ellipses whose equations are given below. (Nos. 1–6.)

1. $4x^2 + y^2 = 12$. *Ans.* $(0, \pm 2\sqrt{3})$, $(0, \pm 3)$, $\sqrt{3}/2$, $4\sqrt{3}$, $2\sqrt{3}$.

2. $x^2 + 4y^2 = 16$.

3. $2x^2 + 5y^2 = 40$. *Ans.* $(\pm 2\sqrt{5}, 0)$, $(\pm 2\sqrt{3}, 0)$, $\sqrt{3/5}$, $4\sqrt{5}$, $4\sqrt{2}$.

4. $7x^2 + 5y^2 = 70$.

5. $4x^2 + 9y^2 = 72$. *Ans.* $(\pm 3\sqrt{2}, 0)$, $(\pm \sqrt{10}, 0)$, $\sqrt{5}/3$, $6\sqrt{2}$, $4\sqrt{2}$.

6. $8x^2 + 3y^2 = 48$.

Find the remaining values a, b, c, e; then write the equation of each ellipse when the given parts are as follows. (Nos. 7–10.)

7. Foci at $(\pm 3, 0)$, $e = 1/2$. *Ans.* $a = 6$, $b = 3\sqrt{3}$, $x^2/36 + y^2/27 = 1$.

8. Center at the origin, minor axis 4 units along the y axis, semi-major axis $\sqrt{5}$ units.

9. Center at the origin, extremities of a latus rectum $(\pm 10/3, 4)$.
Ans. $a = 6$, $b = 2\sqrt{5}$, $c = 4$, $e = 2/3$, $x^2/20 + y^2/36 = 1$.

10. Extremities of minor axes $(\pm 8, 0)$, $e = 3/5$.

From the definition of an ellipse derive its equation if it has the following parts. (Nos. 11–15.)

11. Foci at $(-2, 3)$ and $(6, 3)$, $e = 2/3$.
Ans. $5(x-2)^2 + 9(y-3)^2 = 180$.

12. Vertices at $(-1, 8)$ and $(-1, -4)$, $e = 2/3$.

13. Extremities of the minor axis $(-1, 0)$ and $(-1, -4)$, one focus $(2, -2)$.
Ans. $4(x+1)^2 + 13(y+2)^2 = 52$.

14. One vertex at $(-3, 4)$, nearer focus at $(-1, 4)$, $e = 1/2$.

15. Center at $(-2, 3)$, one focus at $(-2, -1)$, the major axis is twice as long as the minor axis.
Ans. $12(x+2)^2 + 3(y-3)^2 = 64$.

16. Find the equations of the two ellipses with axes parallel to the coordinate axes, respectively, which have the foci and the extremities of the minor axes on the circle $x^2 + y^2 - 2x + 4y - 20 = 0$.

Locate the center, and translate the origin to the center. Find the transformed equation and sketch. (Nos. 17–21.)

17. $4x^2 + y^2 = 8x - 4y$. *Ans.* $(1, -2)$, $4x'^2 + y'^2 = 8$.

18. $x^2 + 4y^2 - 2x - 16y + 1 = 0$.

19. $2x^2 + 3y^2 + 8x - 6y = 1$. *Ans.* $(-2, 1)$, $2x'^2 + 3y'^2 = 12$.

20. $7x^2 + 2y^2 + 7x + 6y + 25/4 = 0$.

21. $x = 3 + 2\sin\theta$, $y = 1 + 3\cos\theta$. *Ans.* $(3, 1)$, $9x'^2 + 4y'^2 = 36$.

22. A line of constant length $a + b$ units has its extremities on the coordinate axes. Find the locus of a point P on this line which is a units from one extremity and b units from the other.

23. The axes of an ellipse coincide with the coordinate axes. Find its equation if it contains the points $(2, 4)$ and $(6, -2)$.
Ans. $3x^2 + 8y^2 = 140$.

24. The same as Problem 23, if it contains the points $(-3, 5)$ and $(4, -1)$.

25. A semi-elliptic arch spans a roadway 150 feet wide. If the center of the arch is 30 feet above the road, what width of the road will have a clearance of at least 20 feet? *Ans.* $50\sqrt{5} = 111.8$ ft.

26. A semi-elliptic arch is to be built over a four-lane highway. It is required that the arch shall be at least 20 feet above the two central sections, each of which is 15 feet wide, and at least 15 feet above the two outside sections, each of which is 10 feet wide. Find the necessary height and span of the arch.

37. The Hyperbola. *The* ***hyperbola*** *is the locus of a point the difference of whose distances from two fixed points is a constant.* Let the two fixed points which are called the ***foci*** be taken on the x axis at F' and F with the origin at the mid-point. Call $2c$ the length

of $F'F$. Assume $P(x, y)$ to be any point on the hyperbola; then we have, by the definition above,

$$F'P - FP = 2\,a,$$

or

$$FP - F'P = 2\,a,$$

where $2\,a$ is the constant difference. Corresponding to these relations, we have the equation

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm\, 2\,a.$$

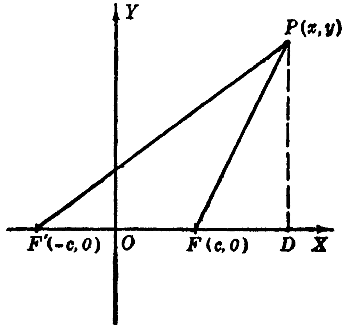


FIG. 62

The rationalized form of this expression is identically the same as the form obtained for the ellipse in which case the sum of the same two radicals is $2\,a$.* That is,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

But in the triangle $F'PF$ the difference of $F'P$ and FP is always less than $F'F$, that is, $2\,a < 2\,c$ or $a < c$. Hence, by writing $-\,b^2$ for $a^2 - c^2$, so that b^2 is positive and b is real, the equation becomes

$$(1) \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

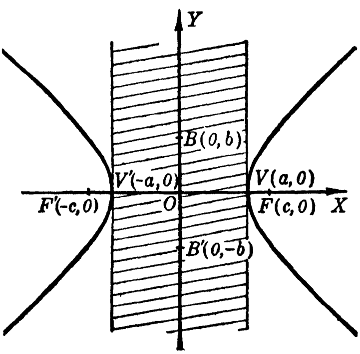


FIG. 63

Equation (1) shows that the hyperbola is symmetric with respect to both the axes and the origin. The x intercepts are $\pm\, a$. There are no y intercepts.

If $x^2 < a^2$, y is imaginary, which means that no part of the curve can lie between the lines $x = \pm\, a$. The segment $V'V$ is called the ***transverse axis,*** V' and V being the ***vertices.*** The segment $B'B$ is called the ***conjugate axis.*** O is the ***center*** of the hyperbola. The ratio c/a

* This is due to the fact that the rationalized form of each of the four radical equations $\sqrt{a} \pm \sqrt{b} = \pm \sqrt{c}$ is obtained by equating to zero the product

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} - \sqrt{b} - \sqrt{c})(-\sqrt{a} + \sqrt{b} - \sqrt{c})(-\sqrt{a} - \sqrt{b} + \sqrt{c}).$$

is called the ***eccentricity*** of the hyperbola; it is denoted by e, and is always greater than 1. The chord through either focus perpendicular to the transverse axis is called a ***latus rectum.***

If the foci are on the y axis the equation is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1. \tag{2}$$

ASYMPTOTES. The hyperbola has two ***asymptotes*** which pass through the center of the curve. *These are lines which the hyperbola approaches so that its distance from them approaches zero as it is indefinitely extended.* The equation of any line through the center is $y = mx$, and its intersections with the hyperbola are found by solving this equation simultaneously with (1). This gives

$$x = \pm \frac{ab}{\sqrt{b^2 - a^2m^2}}, \qquad y = \pm \frac{abm}{\sqrt{b^2 - a^2m^2}}.$$

The intersections will be real or imaginary according as the expression under the radical, $b^2 - a^2m^2$, is positive or negative. However, if $b^2 - a^2m^2 = 0$, that is, if $m = \pm b/a$, the curve approaches these lines but has no finite intersection with them. To prove this statement, let (x_1, y_1) be a point on either line $y = \pm (b/a)x$, and let (x_1, y_2) be a point on the hyperbola with the same abscissa. Then

$$y_1 = \pm \frac{b}{a} x_1, \qquad y_2 = \pm \frac{b}{a}\sqrt{x_1^2 - a^2}.$$

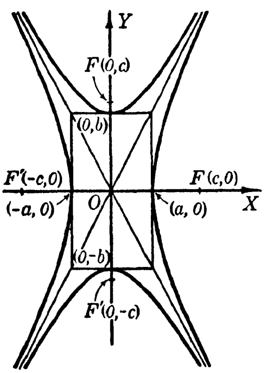


FIG. 64

The difference of these ordinates is

$$y_1 - y_2 = \pm \frac{b}{a}\left(x_1 - \sqrt{x_1^2 - a^2}\right)$$
$$= \pm \frac{ab}{x_1 + \sqrt{x_1^2 - a^2}}.$$

Hence, as x_1 increases indefinitely, y_2 approaches y_1, since $y_1 - y_2$ approaches 0, and the hyperbola approaches the lines $y = \pm (b/a)x$ as asymptotes.

The two curves

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1 \tag{3}$$

are called ***conjugate hyperbolas.*** They have the transverse and

conjugate axes interchanged, have the same asymptotes, a common center, and foci at the same distance from the center. The eccentricities are $e_1 = c/a$ and $e_2 = c/b$, respectively.

Draw the rectangle $x = \pm a$, $y = \pm b$. Its diagonals are the asymptotes to each of the conjugate hyperbolas (3). The circle circumscribing this rectangle intersects the axes in the foci for each curve.

38. Equilateral Hyperbolas. If $a = b$, the two conjugate hyperbolas have the same shape and eccentricity. The asymptotes are the lines $x \pm y = 0$ and the equations of the two curves become

$$(1) \qquad x^2 - y^2 = \pm a^2.$$

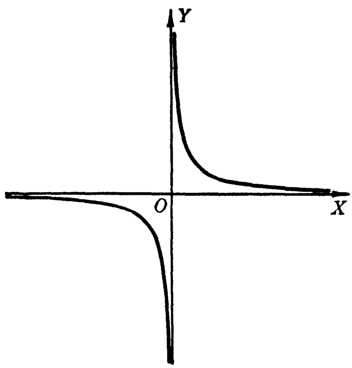


FIG. 65

These hyperbolas are called ***equilateral*** or ***rectangular hyperbolas.***

A more important form of the equation of the equilateral hyperbola is obtained by taking the asymptotes as the coordinate axes. This equation may be obtained from (1) by using the equations of transformation by rotation (§ 22), for $\theta = 45°$. These equations are

$$x = (\sqrt{2}/2)(x' - y'), \qquad y = (\sqrt{2}/2)(x' + y').$$

Substituting in (1) and simplifying, we find the result

$$(2) \qquad x'y' = \pm \frac{a^2}{2}.$$

39. Other Forms of the Equation of the Hyperbola. The following forms of the equation of the hyperbola are important:

$$(1) \qquad Ax^2 + By^2 + Dx + Ey + F = 0,$$

where A and B are of unlike sign. By completing the squares, and translating the origin to the center, (1) becomes

$$(2) \qquad Ax'^2 + By'^2 = C',$$

where A and B are the same as in (1). This hyperbola has its transverse axis along the x' axis or y' axis according as C' has the

same sign as A or as B. If $C' = 0$, then (2) is the product of linear factors, and the graph is a pair of straight lines.

The equation

$$(3) \qquad y = \frac{ax + b}{cx + d},$$

where a, b, c, d have any values, is an equilateral hyperbola. It is evident that $cx + d = 0$ or $x = -d/c$ is a vertical asymptote. Solving (3) for x, we get

$$x = -\frac{dy - b}{cy - a},$$

from which we see that $y = a/c$ is a horizontal asymptote. If we translate the origin to the intersection of these asymptotes $(-d/c, a/c)$ by writing $(x' - d/c)$ and $(y' + a/c)$ for x and y, respectively, equation (3) reduces to

$$(4) \qquad x'y' = k,$$

where $k = (bc - ad)/c^2$.

It can be shown that the asymptotes of the hyperbola in equation (2) are the linear factors of $Ax'^2 + By'^2 = 0$. This brings us to a useful converse theorem which is stated without proof.

If L_1 and L_2 are any two linear expressions in x and y, and k has any value except zero, then $L_1 \cdot L_2 = k$ is some hyperbola whose asymptotes are $L_1 = 0$ and $L_2 = 0$.

EXAMPLES

1. Given the hyperbola $9x^2 - 16y^2 + 100 = 0$, find the coordinates of the foci and vertices, the eccentricity, and the equations of the asymptotes and of the conjugate hyperbola.

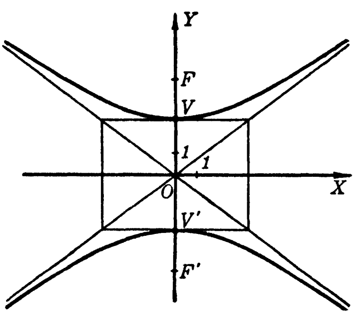


FIG. 66

SOLUTION. Write the equation in the form

$$\frac{y^2}{\left(\frac{5}{2}\right)^2} - \frac{x^2}{\left(\frac{10}{3}\right)^2} = 1.$$

Here the transverse axis along the y axis is of length 5, the vertices being $(0, \pm 5/2)$; $c = \sqrt{a^2 + b^2} = 25/6$, and the foci are $(0, \pm 25/6)$; $e = 5/3$. The slopes of the asymptotes are $\pm 3/4$; hence the equations of the asymptotes are

$$y = \pm \frac{3}{4}x.$$

The conjugate hyperbola is

$$\frac{x^2}{\left(\frac{10}{3}\right)^2} - \frac{y^2}{\left(\frac{5}{2}\right)^2} = 1.$$

2. Find the equation of the hyperbola with foci at $(2, -1)$ and $(-6, -1)$ with $e = 2$.

SOLUTION. The center is $(-2, -1)$. The transverse axis is along $y = -1$; $e = c/a = 2$ and $c = 4$. Hence $a = 2$ and the vertices are $(0, -1)$, and $(-4, -1)$. Let $P(x, y)$ be any point on the hyperbola. Then we must have, from the definition, $F'P - FP = \pm 2a$, or

$$\sqrt{(x+6)^2 + (y+1)^2} - \sqrt{(x-2)^2 + (y+1)^2} = \pm 4.$$

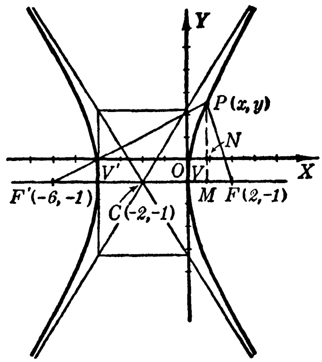


FIG. 67

Rationalizing this equation, we obtain the equation

$$3x^2 - y^2 + 12x - 2y - 1 = 0.$$

PROBLEMS

Find the vertices and foci, the eccentricity, the equations of the asymptotes and sketch each of the following hyperbolas. (Nos. 1–8.)

1. $3x^2 - 2y^2 + 24 = 0$.
Ans. $(0, \pm 2\sqrt{3})$, $(0, \pm 2\sqrt{5})$, $\sqrt{5/3}$, $x\sqrt{3} \pm y\sqrt{2} = 0$.

2. $4x^2 - y^2 = 16$.

3. $9x^2 - 16y^2 = 144$. *Ans.* $(\pm 4, 0)$, $(\pm 5, 0)$, $5/4$, $3x \pm 4y = 0$.

4. $3x^2 - 4y^2 + 60 = 0$.

5. $5y^2 - 3x^2 = 30$.
Ans. $(0, \pm\sqrt{6})$, $(0, \pm 4)$, $2\sqrt{6}/3$, $x\sqrt{3} \pm y\sqrt{5} = 0$.

6. $9x^2 - 4y^2 = 49$.

7. $3y^2 - 7x^2 = 24$.
Ans. $(0, \pm 2\sqrt{2})$, $(0, \pm 4\sqrt{5/7})$, $\sqrt{10/7}$, $x\sqrt{7} \pm y\sqrt{3} = 0$.

8. $7x^2 - 3y^2 = 84$.

Find the remaining values a, b, c, e; then write the equation of the hyperbola with the given parts below. (Nos. 9–12.)

9. Center at the origin, a focus at $(5, 0)$, $e = 2$.
Ans. $a = 5/2$, $b = 5\sqrt{3}/2$, $12x^2 - 4y^2 = 75$.

10. Extremities of the conjugate axis $(\pm 5, 0)$, $e = 3/2$.

11. Foci at $(0, \pm 6)$, slope of asymptotes $\pm 4/3$.

Ans. $a = 24/5$, $b = 18/5$, $e = 5/4$, $(5\,y/4)^2 - (5\,x/3)^2 = 36$.

12. Center at the origin, extremities of a latus rectum at $(\pm 14/3, 8)$.

Derive the equation of each hyperbola in the following cases. (Nos. 13–16.)

13. Center at $(-2, 2)$, one focus $(-2, 7)$, one vertex $(-2, -1)$.

Ans. $(x + 2)^2/16 - (y - 2)^2/9 + 1 = 0$.

14. Foci at $(3, 5)$ and $(13, 5)$, $e = 5/4$.

15. Extremities of the conjugate axis $(-1, 2)$ and $(-1, -6)$, $e = \sqrt{2}$.

Ans. $x^2 - y^2 + 2\,x - 4\,y = 19$.

16. Vertices at $(3, 2)$ and $(3, -4)$, length of latus rectum $32/3$ units.

17. Find the equation of the hyperbola with center at the origin, foci on the x axis, and passing through the points $(5, 9/4)$ and $(4\sqrt{2}, 3)$.

Ans. $x^2/16 - y^2/9 = 1$.

18. Two vertices of a triangle are fixed at $(\pm a, 0)$. Find the equation of the locus of the third vertex if the product of the slopes of the variable sides is b^2/a^2.

19. Find the locus of a point whose distance from $(3, 4)$ is $3/2$ its distance from the line $x = 1$. *Ans.* $5\,x^2 - 4\,y^2 + 6\,x + 32\,y = 91$.

20. Find the locus of a point whose distance from $(-2, -4)$ is twice as great as its distance from the line $y + 1 = 0$.

Find the center of each of the following hyperbolas. Translate the origin to the center and find the eccentricity. (Nos. 21–24.)

21. $5\,x^2 - 2\,y^2 + 20\,x - 4\,y - 18 = 0$. *Ans.* $(-2, -1)$, $\sqrt{7/2}$.

22. $3\,x^2 - 3\,y^2 - 4\,x + 8\,y = 31$.

23. $3\,x^2 - 2\,y^2 + 12\,x + 4\,y + 20 = 0$. *Ans.* $(-2, 1)$, $\sqrt{5/3}$.

24. $4\,x^2 - 8\,y^2 + 4\,x + 32\,y + 1 = 0$.

Translate the origin to the center and find the transformed equation. (Nos. 25–26.)

25. $x = (5\,y + 8)/(4\,y - 1)$. *Ans.* $(5/4, 1/4)$, $16\,x'y' = 37$.

26. $y = (3\,x - 5)/(2\,x + 7)$.

27. One vertex of a hyperbola is at $(1, 3)$, the corresponding focus at $(1, 6)$. If the slope of one asymptote is $3/4$, find the center and eccentricity.

Ans. $(1, -1\tfrac{1}{2})$, $5/3$.

28. Given the equilateral hyperbola $x^2 - y^2 = a^2$. Prove that the distance of any point on the curve from the center is the mean proportional between its distances from the foci.

29. Change $9\,x^2 - 4\,y^2 + 36 = 0$ to parametric form if $y = 3 \sec \theta$ is to be one equation.

30. Sketch and name $x = 3 \tan t$, $y - 2 = 4 \sec t$. Transform to its rectangular equation.

40. Conics. The parabola, ellipse, and hyperbola are included in a class of curves called ***conics*** or ***conic sections.*** A conic may be defined as *the locus of a point such that its distance from a fixed point is in a constant ratio to its distance from a fixed line.* The constant ratio is the eccentricity e, and the curve is an ellipse, parabola, or hyperbola according as e is less than, equal to, or greater than 1. The fixed point is ***a focus*** and the fixed line ***a directrix.*** To derive the equation of the conic in polar coordinates, *let the given focus be the pole* and *let the directrix be perpendicular to the polar axis,* and at a distance p from the focus. Then if $P(r, \theta)$ is any point on the locus, by the definition we have $OP/NP = e$, that is, $OP = e \cdot NP$. Hence $r = e(p + r \cos \theta)$, from which we find

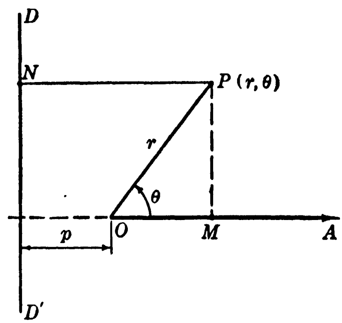


FIG. 68

$$r = \frac{ep}{1 - e\cos\theta}. \tag{1}$$

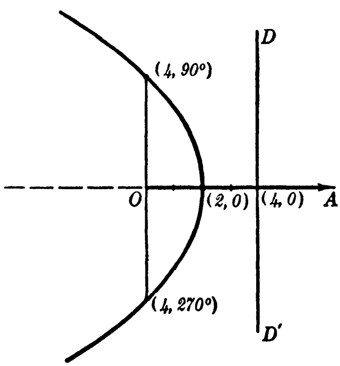


FIG. 69

In general the equations

$$r = \frac{\pm ep}{1 \pm e\cos\theta},$$

$$r = \frac{\pm ep}{1 \pm e\sin\theta}$$

represent conics. The latter equations are the forms obtained when the directrix is parallel to the polar axis.

EXAMPLES

1. Plot the graph of $r(1 + \cos\theta) = 4$.

SOLUTION. This is a parabola since $e = 1$. Then $p = 4$. Some pairs of values are:

θ	0°	60°	90°	120°	180°	240°	270°	300°	360°
r	2	8/3	4	8	∞	8	4	8/3	2

from which the curve can be drawn (Fig. 69).

2. Draw the graph of $r(3 - 2 \sin \theta) + 6 = 0$.

SOLUTION. This equation may be written

$$r = \frac{-2}{1 - \frac{2}{3}\sin\theta} = \frac{-(2/3)3}{1 - \frac{2}{3}\sin\theta}.$$

Hence $e = 2/3$, $p = 3$, and the locus is an ellipse (Fig. 70).

θ	0°	30°	90°	150°	180°	210°	270°	330°	360°
r	-2	-3	-6	-3	-2	$-3/2$	$-6/5$	$-3/2$	-2

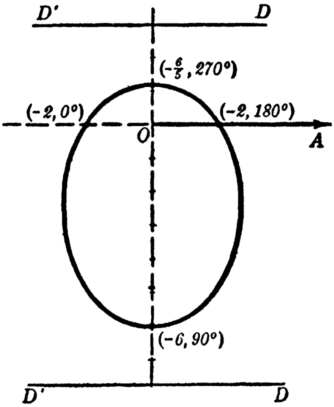


FIG. 70

Let the student show that the graph of $r(3 + 2 \sin \theta) = 6$ gives the same ellipse. Explain

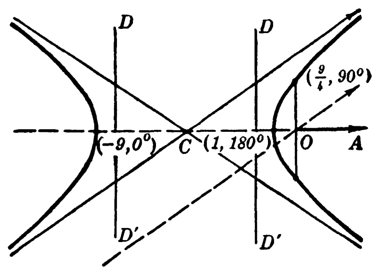


FIG. 71

3. Draw the graph of $r(4 - 5 \cos \theta) = 9$.

SOLUTION. Write the equation

$$r = \frac{\frac{9}{4}}{1 - \frac{5}{4}\cos\theta} = \frac{\left(\frac{5}{4}\right)\left(\frac{9}{5}\right)}{1 - \frac{5}{4}\cos\theta}.$$

Hence $e = 5/4$, $p = 9/5$. The curve is a hyperbola. We observe that the values of θ which make $1 - (5/4) \cos \theta = 0$ will make r become infinite. These values of θ give the lines through the pole which are parallel, respectively, to the asymptotes.

θ	0°	60°	90°	120°	180°	240°	270°	300°	360°
r	-9	6	9/4	18/13	1	18/13	9/4	6	-9

Note that the transverse axis between $(-9, 0°)$ and $(1, 180°)$ is of length 8.

PROBLEMS

Plot the graph of each of the following equations. (Nos. 1–8.)

1. $2r = 4 + r \sin \theta$.

2. $3r + 4 = 5r \sin \theta$.

3. $r = 2 + r \cos \theta$.

4. $r = 3 - 2r \sin \theta$.

5. $r(4 - 4 \sin \theta) = 9$.

6. $r(4 \cos \theta - 3) = 9$.

7. $r(2 \cos \theta - 5) + 6 = 0$.

8. $r(1 + \cos \theta) = 5$.

Derive the equation of each of the following curves. (Nos. 9–12.)

9. The circle of radius 5 units tangent to $\theta = 0$ at the pole.

Ans. $r = \pm 10 \sin \theta$.

10. The ellipse with $e = 1/2$, $r \sin \theta + 2 = 0$ as directrix, and focus at the pole.

11. The hyperbola with $e = 3/2$ and a line 10 units from the pole perpendicular to the polar axis as directrix, and focus at the pole. (*Two cases.*)

Ans. $2r = 3(10 \pm r \cos \theta)$.

12. The same as Problem 11 except the directrix is parallel to the polar axis.

13. Transform $x^2 = y(2a - y)$ to polar coordinates. Name and draw its graph. *Ans.* $r = 2a \sin \theta$.

14. Discuss $r(1 - e \cos \theta) = ep$ with reference to symmetry, asymptotes, closed or open form, for $e = 1, 2, 0.2, 3, 0.3$.

15. The same as Problem 14 for $r(1 + e \sin \theta) = ep$, $e = 0.5, 1, 2$.

16. Find the locus of P if its distance from the pole is 2/3 of its distance from $r \cos \theta + 3 = 0$.

41. Cycloid. Involute of a Circle. These curves have some important applications and their equations are usually given in parametric form. *The cycloid is the path traced by a point on the circumference of a circle as it rolls along a straight line.* Let C be the center of a circle of radius a and let $P(x, y)$ be any point on the circumference.

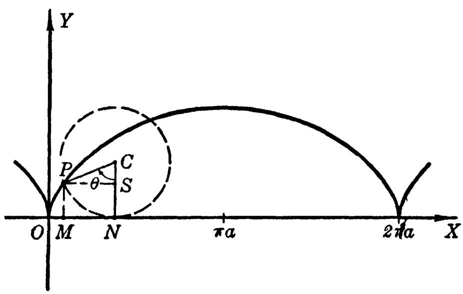


FIG. 72

To find the locus of P as the circle rolls along the x axis, let O, the point where P is in contact with the line, be the origin. We can express the coordinates of P in terms of the angle θ which the radius CP makes with the vertical line CN. Thus

$$x = OM = ON - MN = \text{arc } NP - PS = a\theta - a \sin \theta,$$

and

$$y = MP = NC - SC = a - a \cos \theta.$$

Hence

$$(1) \qquad \boldsymbol{x = a(\theta - \sin \theta), \qquad y = a(1 - \cos \theta).}$$

Each value for θ in (1) fixes a value for x and a value for y and hence a point on the cycloid.

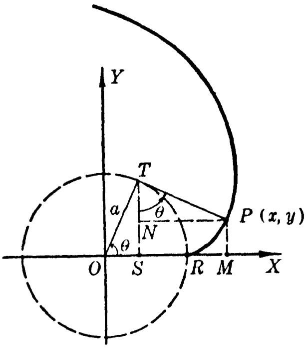


FIG. 73

If a string is wound about a fixed circle, the path traced by any point of the string as it is kept taut and unwound from the circle is called an ***involute of the circle.***

Let O be the center of the fixed circle and P be the point of the string which meets the circle at R on the x axis. Let T be the point of tangency corresponding to the point P and call θ the angle between the radii OT and OR. Then

$$TP = \text{arc } RT = a\theta.$$

If we draw TS perpendicular to OR and NP perpendicular to ST, we find

$$x = OM = OS + NP = a \cos \theta + a\theta \sin \theta,$$

and

$$y = MP = ST - NT = a \sin \theta - a\theta \cos \theta.$$

Therefore, the involute of the circle has the equations

$$(2) \qquad \begin{cases} x = a(\cos \theta + \theta \sin \theta), \\ y = a(\sin \theta - \theta \cos \theta). \end{cases}$$

PROBLEMS

Draw the graph of each of the following pairs of parametric equations. (Nos. 1–3.)

1. $x = 2(\theta - \sin \theta)$, $y = 2(1 - \cos \theta)$.

2. $x = 1 - \cos \theta$, $y = \theta - \sin \theta$.

3. $x = 3(\theta + \sin \theta)$, $y = 3(1 - \cos \theta)$.

4. Sketch a section of an involute of a circle with a radius of 3 units.

42. Empirical Equations. Often the exact form of an equation is not known, the only information obtainable being a table of corresponding values of the related variables. Even then, the values of the variables are inexact if they are obtained by measurement. In that case, the problem is to find a formula or relation between the variables which the given values satisfy approximately. There are six such formulas, with two constants, generally

used for *fitting* a curve to the points determined by the given data. By ***fitting*** we mean the constants may be determined so that the graph of the formula will come as near the plotted points of the given data as the accuracy of the observations demands.

43. Two-Constant Formulas. The formulas with two constants commonly used are:

(1) $y = mx + b$ (*straight-line formula*),

(2) $y = ax^2 + b$ (*parabolic formula*),

(3) $y = ax^n$ (*power formula*),

(4) $y = ae^{bx}$, or $y = ab^x$ (*exponential formula*),

(5) $xy = ax + b$ (*hyperbolic formula*),

(6) $xy = ax + by$ (*hyperbolic formula*).

We determine which formula to use as follows:

(*a*) If the table of data is plotted and the curve suggested by the points is a straight line, assume formula (1).

(*b*) Let $x^2 = z$ and (2) becomes a straight-line formula with each point located at (z, y).

(*c*) Taking the logarithm of each member of (3), we have

$$\log y = \log a + n \log x,$$

which is linear in $\log y$ and $\log x$. Hence, if the points plotted from the logarithms of the given data suggest a straight line, assume formula (3). Either natural or common logarithms may be used.

(*d*) Treating formula (4) in like manner, we have

$$\log y = \log a + bx \log e, \qquad \text{or} \qquad \log y = \log a + x \log b,$$

which are linear in x and $\log y$. Hence, if the points plotted with the given values of one variable as abscissas and the logarithms of the values of the other as ordinates suggest a straight line, assume formula (4).

(*e*) Setting $xy = z$, we see that formula (5) takes the form $z = ax + b$ and that it is linear in x and z. Here we plot the given data with the ordinates replaced by the products of corresponding variable values and, if the points suggest a straight line, we assume formula (5).

(*f*) Formula (6) is usually written

$$\frac{a}{y} + \frac{b}{x} = 1,$$

and may be used if $1/x$ and $1/y$ suggest a straight-line formula.

44. Methods of Determining Constants. (*a*) GRAPHICAL METHOD. Plot the points determined by the given data. In so doing, choose as coordinates either (x, y) or (x^2, y) or $(\log x, \log y)$, or $(x, \log y)$, or (x, xy), or $(1/x, 1/y)$, where x and y are corresponding values of the given data.

If the formula is one of the two-constant formulas of § 43, one set of points will suggest a straight line. Draw a straight line fitting the points and substitute the coordinates of two points on the line in the formula chosen to get two equations involving the unknown constants. These are solved simultaneously for the constants.

In constructing the straight line, it is desirable to have as many of the plotted points as near the line as possible, as well as the same number on each side of it. Of course, the two points used to determine the constants have their coordinates *read* from the graph and consequently are inaccurate. However, better results are obtained if the two points used to determine the constants are chosen as far apart as possible in the group of plotted points.

(*b*) METHOD OF AVERAGES. A method which takes into account all the data, and not merely two selected points which may not even occur in the given observations, is that of ***averages.*** In using this method, we must determine the formula to be used as in the previous method, but, having fixed on the formula, the problem is arithmetical. Proceed as follows: Substitute each pair of values used in plotting the points in the linear formula assumed. That gives as many equations as there are pairs of corresponding values. Then divide these equations into two groups as nearly equal in number as possible. Add corresponding members of the equations of each group, thus obtaining two equations to determine the two constants. Solve these equations simultaneously and make the proper substitutions in the original formula.

If the form of the desired formula is known from the nature of the problem or is given together with the data, the method of averages does not depend upon any graphical observations. In any case, this method is generally the better to apply because it

takes into account all given data and does not depend upon reading fractional values of coordinates. However, if the data are not necessarily of equal validity, the graph will suggest the pairs of values which are questionable and probably negligible.

45. A More General Parabolic Formula. The parabolic formula

$$y = ax^2 + bx + c \tag{1}$$

occurs occasionally. The method of averages may be used at once. Merely divide the observational equations into three groups, add corresponding members of the equations of each group, and solve the three equations for a, b, and c.

EXAMPLE

Find a two-constant formula for the following data:

x	0	2.1	5.6	9.3	11.5
y	20	18.92	17.34	15.8	14.96

SOLUTION. We find from trial that points with coordinates x and $\log_{10} y$ fit a straight line. The table of values for those points and the corresponding graph are given below.

x	$\log_{10} y$
0	1.3010
2.1	1.2769
5.6	1.2390
9.3	1.1987
11.5	1.1749

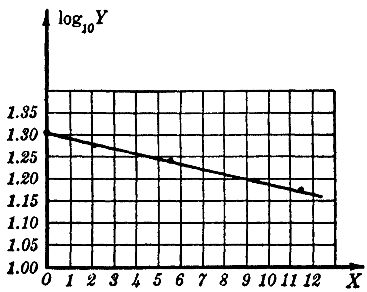


FIG. 74

(a) GRAPHICAL METHOD. Draw a straight line fitting the points plotted. The straight line seems to pass through the points (0, 1.3) and (11.5, 1.17). One recognizes the absurdity of attempting to read or plot the second point chosen for the assumed scale. However, using the selected points, we have the two equations from

$$\log_{10} y = \log_{10} a + bx \log_{10} e$$

as follows:

$$1.3 = \log_{10} a, \qquad 1.17 = \log_{10} a + b(11.5)(0.4343).$$

Solving these, we find

$$\log_{10} a = 1.3, \qquad b = -0.0260,$$

whence $a = 19.95$, and the desired formula is

$$y = 19.95\, e^{-0.0260x}.$$

(b) METHOD OF AVERAGES. Substitute in

$$\log_{10} y = \log_{10} a + bx \log_{10} e.$$

Then

$$1.3010 = \log_{10} a + 0,$$

$$1.2769 = \log_{10} a + 2.1(0.4343)b.$$

Adding, we find

$$2.5779 = 2 \log_{10} a + 0.9120\, b.$$

Also

$$1.2390 = \log_{10} a + 5.6(0.4343)b,$$

$$1.1987 = \log_{10} a + 9.3(0.4343)b,$$

$$1.1749 = \log_{10} a + 11.5(0.4343)b.$$

Adding, we have

$$3.6126 = 3 \log_{10} a + 11.4655\, b.$$

Solving the equations derived by additions, we find

$$\log_{10} a = 1.3004, \qquad b = -0.0252,$$

whence $a = 19.97$, and the desired formula is

$$y = 19.97\, e^{-0.0252x}.$$

PROBLEMS

The following sets of data satisfy approximately the given formula. Find the laws.

1.

x	0.5	1 0	1.5	2 0	2.5	3 0
y	0 31	0.81	1.29	1.85	2 51	3.02

for $y = ax + b$. *Ans.* $y = 1.1\, x - 0.30$.

2.

x	10	20	30	40	50	60
y	3.26	4.73	6.24	7.49	9.01	10.51

for $y = mx + b$.

3.

x	6.0	6.9	7.5	8.7
y	7.5	11.5	13.8	20.5

for $y = a + bx^2$. *Ans.* $y = 0.5\, x^2 - 10.6$.

4.

x	2.0	4.7	7.1	8.4
y	75.6	55.7	24.5	2.4

for $y = a + bx^2$ or $y = a \cdot b^x$.

5.

x	3.0	4.1	5.3	7.0
y	1 90	5 75	11 80	24.10

for $y = ax^2 + bx$. *Ans.* $y = 0.70\,x^2 - 1.47\,x$.

6.

x	0	5	10	15	20	30
y	0	5.0	6.8	7.4	8.0	8.7

for $y = ax^2 + b$ or $x = axy + by$.

7.

x	5	10	15	20	25	30
y	20.0	24 3	28 3	32.1	35 6	39.0

for $y = ax^2 + bx + c$. *Ans.* $y = -0.004\,x^2 + 0.9\,x + 15.5$.

8.

x	0.5	2 0	3 5	4.5
y	0.1	0 9	2 2	3.3

for $y = ax^n$.

9.

x	1 0	1 5	2 0	2 5	3.0	3.5
y	1 0	4 1	8 5	14 1	21 0	29 1

for $y = ax^2 + b$. *Ans.* $a = 2.5$, $b = -1.5$.

10.

u	26.4	22.4	19 1	16.3	14 0
p	14 7	17 5	20 8	24 5	28.8

for $pu^n = c$.

11.

x	1.22	0 426	0.047	0.005
y	0 676	0 074	0.004	0

for $y = ax^n$. *Ans.* $a = 0.40$, $n = 1.78$.

12.

x	1.124	0.342	0.511	0.730
y	1.002	0.604	0.494	0.414

for $y = a(y/x)^n$.

13.

x	5	10	15	20	25	30
y	6.1	6.8	7.4	8 0	8.5	9.2

for $y = x/(ax + b)$. *Ans.* $a = 0.1$, $b = 0.5$.

14.

d	1.5	3 0	4.0	6.0
B	13.43	75.13	152 51	409.54

for $B = a \cdot d^n$.

15.

x	1	0.50	0.25	0.17	0.10
y	0.77	0.45	0.34	0.20	0.16

for $xy = ax + by$. *Ans.* $xy = 1.49\,x - 0.09\,y$.

16.

x	1.3	3.4	6.2	8.3
y	0 56	0 91	1.11	1 18

for $xy = ax + by$ or $y = ax^n$.

17. An iron plate with one straight edge has its width given by d at intervals x along the straight edge. Find a law for the other edge if we have

x	0	4	8	12	16	20
d	2	4	5	6	9	12

Ans. $d = 0.006(x - 8)^3 + 4.81$.

ADDITIONAL PROBLEMS

1. Find the locus of a point P if the difference of its distances from the fixed points $(0, \pm c)$ is a constant $2a$. *Ans.* A hyperbola.

2. Write the general equation of a line through P_1 so that it will have only one arbitrary constant.

3. Write the general equation of the line through P_1 and parallel to $ax + by + c = 0$. *Ans.* $a(x - x_1) + b(y - y_1) = 0$.

4. The same as Problem 3 except perpendicular to the given line.

In Problems 5–8, find the equation of each of the lines described below.

5. Through the origin and making an angle of 45° with $x = 2y + 3$. *Ans.* $y = 3x$.

6. Through $(4, -6)$ and perpendicular to $6x - 7y + 1 = 0$.

7. Through $(-1, -1)$ and perpendicular to the line through that point and $(2, 3)$. *Ans.* $3x + 4y + 7 = 0$.

8. Through $(3, 5)$ and parallel to the line through $(2, 5)$ and $(-5, -2)$.

9. Find the distance from $x + 2y = 5$ to $(2, -6)$. *Ans.* $3\sqrt{5}$ units.

10. Given $A(2, -5)$ and P_1. Find the equation of AP_1, its length, its slope, and its mid-point.

11. Find the equation of the circle with its center at the point $(-2, -2)$ which is tangent to $x + y = 6$. *Ans.* $(x + 2)^2 + (y + 2)^2 = 50$.

12. Find the equation of the circle through $(3, 4)$ and tangent to the x axis at $(-1, 0)$.

13. The channel of a river for some distance remains equidistant from a rock in the stream and a straight shore line. If the rock is 600 feet from the shore, derive an equation for this part of the channel. *Ans.* $y^2 = 1200x$.

14. Find the circle through the vertex and focus of $x^2 = 8y$ with its center on $x - y - 2 = 0$.

15. Find the circle whose diameter is the chord of $y^2 = 1 - x$ cut off on $x - y + 1 = 0$. *Ans.* $(2x + 3)^2 + (2y + 1)^2 = 18$.

16. Find the equation of the parabola with its focus at the center of the circle $3x^2 + 3y^2 + 12x - 6y = 10$ if its directrix is parallel to the x axis and tangent to the circle. (Two cases.)

17. Find the locus of a point whose distance from (0, 9) is twice its distance from $y = 4$. *Ans.* $x^2 - 3y^2 + 14y + 17 = 0$.

18. A parabolic arch of 100 ft. base has 12 ft. clearance 6 ft. from one end of the base. What clearance has the arch at its top?

19. A parabolic arch is 10 ft. high and 15 ft. wide at its base. How far from the end of the base is the clearance 6 ft.? *Ans.* 4.74 ft.

20. Find the equation of the circle which passes through the vertex and the extremities of the latus rectum of the parabola $x^2 + 10y = 0$.

21. If two vertices of a triangle are fixed at $(\pm a, 0)$, find the locus of the third vertex if the product of tangents of the base angles is b^2/a^2.

Ans. Ellipse.

22. Using the definition of the ellipse, derive its equation if the foci are (2, 2), (6, − 3) and it passes through (2, 0). What is the value of e?

23. Name each curve, sketch, and transform to rectangular representation:

(*a*) $r(1 - \sin\theta) = 3$; (*b*) $r(3 + 2\sin\theta) = 5$; (*c*) $r(2 - 3\cos\theta) = 5$.

Ans. (*a*) Parabola, $x^2 = 6y + 9$.

24. Two lighthouses are 5 miles apart. If a distress signal is heard at one lighthouse 15 seconds before it is heard at the other, what path should a vessel take to locate the signal? (Assume sound travels 1 mile in 5 seconds.)

25. Which law, $xy = ax + b$, $y^2 = ax + b$, or $xy = ax + by$, is best adapted for the following data?

x	8.05	7.54	5.16	3.22	1.57
y	1.20	1.41	2.15	2.59	2.91

Ans. $y^2 = 10.3 - 1.1x$.

26. The no-load magnetization curve of a direct-current generator taken at 1200 r.p.m. was found by test to include the following points:

I	0	0.1	0.2	0.3	0.4	0.5	0.6
E	4.4	29.6	52.6	73.1	91.5	105.3	115.1

I	0.8	1.0	1.2	1.4	1.6	2.0
E	126.4	132.1	134.7	135.9	136.4	137.1

where I is field current in amperes and E is no-load voltage. Find the empirical equation for this curve:

(*a*) Using Froelich's equation $E = aI/(b + I)$.
(*b*) Using a modification of Froelich's equation $E = aI/(b + I) + c$.
(*c*) Using a power series $I = a + bE + cE^2 \cdots$.

27. Derive equation (6), § 28, directly from a figure.

CHAPTER III

THE DERIVATIVE

46. Constants. Variables. A quantity which has a fixed value is called a ***constant.*** These are of two kinds. An ***absolute constant*** is a fixed number, as $2, -3/2, \sqrt{5}, \pi, \log_{10} 17, \sin 24° 15'$. An ***arbitrary constant*** is one which is represented by some letter, as a, c, k, m. Such a constant is assumed to have a definite value which it retains throughout a particular problem.

A ***variable*** is a quantity which may have different values in a given problem. Thus the temperature of a certain object is a variable quantity. The length of a chord of a circle of radius 10 is a variable which may have any value between 0 and 20. A variable is usually represented by some letter in the latter part of the alphabet, as x, y, u, v.

47. Functions. *A* ***function*** *of a given variable x is another variable quantity which has one or more definite values corresponding to each value assigned to the variable x.* To illustrate, $2\sqrt{25 - x^2}$ is a function of the variable x since it is also a variable quantity, and depends for its value on the value assigned to x. A function of x is commonly represented by the symbol $f(x)$, which is read "f of x," or by a single letter, as y. Then x is called the ***independent variable*** and $f(x)$, or y, is called the ***dependent variable*** or ***function of x.*** In general we speak of x simply as ***the variable*** and of $f(x)$, or y, as ***the function.*** We may refer to the function mentioned above either by $f(x) = 2\sqrt{25 - x^2}$ or $y = 2\sqrt{25 - x^2}$, where $f(x)$ or y are merely other symbols for the expression $2\sqrt{25 - x^2}$. This function has the following geometric interpretation: In a circle of radius 5 units, if any chord is drawn at a distance of x from the center, then the length of the chord, y, is $2\sqrt{25 - x^2}$. Hence we can say that this function expresses the length of a chord of the given circle in terms of its distance from the center of the circle.

A function of a variable may be represented (a) by an equation connecting the variable and the function, and (b) by a graph in which the corresponding values of the variable and function are

the abscissas and ordinates, respectively. If a function is given by a formula or equation, the corresponding pairs of values may be obtained from it and the graph constructed. However, two physical quantities may be related without being connected by a known formula. Thus, the temperature at a given place may be a function of the time. A table of values of time and temperature can be recorded by observation and the results plotted to form a graph, but it is impossible to write a formula expressing the temperature in terms of the time, or to calculate the temperature at some future time.

Since $y = f(x)$ means that y is a function of x, in a similar manner $z = f(u, v)$ means that z is a function of the two variables u and v. To distinguish between different functions in the same discussion, different letters are used, as $f(x)$, $g(x)$, $\phi(x)$, $\psi(x)$. Throughout the same discussion, however, the same symbol refers always to the same function. Thus if we have given $y = f(x) = 2x - \sqrt{9 + x^2}$, then $f(4)$ means the value of $f(x)$ when 4 is substituted for x. That is, $f(4) = 3$; similarly $f(0) = -3$, $f(-2) = -4 - \sqrt{13}$, $f(a) = 2a - \sqrt{9 + a^2}$.

48. Inverse Functions. If $y = f(x)$, then any value assigned to y will determine one or more corresponding values of x. That is, x is also a function of y, or $x = \phi(y)$; then $\phi(y)$ is called the ***inverse function*** of $f(x)$. To get $\phi(y)$, solve the equation $y = f(x)$ for x. There are many cases where this cannot be done by means of elementary algebra, but the relation $x = \phi(y)$ in general exists. Examples of inverse functions are

$$y = x^2 + 2x - 1 \quad \text{and} \quad x = -1 \pm \sqrt{2 + y};$$

$$y = 3 \log_a (x/2) \quad \text{and} \quad x = 2a^{y/3};$$

$$y = (1/3) \sin (2x) \quad \text{and} \quad x = (\sin^{-1} 3y)/2.$$

49. Explicit and Implicit Functions. In the form $y = f(x)$ we call y an ***explicit function*** of x since y is given explicitly in terms of x. However, if two variables x and y are connected by a relation of the type $\phi(x, y) = 0$, then y is called an ***implicit function*** of x, since the existence of this relation implies that y is a function of x. Likewise, x is an ***implicit function*** of y. Solving $\phi(x, y) = 0$ for either variable gives an explicit function in terms of the other. For example,

$$\phi(x, y) = x^2 + 2xy - y^2 + 4 = 0$$

gives x and y implicitly. Solving for each variable in turn, we obtain the explicit functions $y = x \pm \sqrt{2x^2 + 4}$, and its inverse, $x = -y \pm \sqrt{2y^2 - 4}$.

50. Types of Functions. The functions which are used in an elementary course of the calculus are *algebraic functions*, and certain ***transcendental functions*** including *logarithmic functions*, *inverse logarithmic* or *exponential functions*, *trigonometric functions*, and *inverse trigonometric functions*. These are the functions which are of fundamental importance also in a study of the physical and engineering sciences.

PROBLEMS

1. Express the area of a square inscribed in a circle as a function of its radius. *Ans.* $A = 2r^2$.

2. Express the volume of a right circular cone whose altitude is one-half the radius of its base as a function of the altitude.

3. Express the volume of a right circular cylinder whose altitude is equal to its diameter of a base as a function of the radius of the base. *Ans.* $V = 2\pi r^3$.

4. Express the surface of a cylinder of volume V cubic units in terms of the radius of the base.

5. Express the volume of a vessel made of a cylinder with hemispheres on each end as a function of the length of the vessel, if the length of the cylindrical part is three times the radius of the ends. *Ans.* $V = 13\pi l^3/375$.

6. Express the volume of a sphere as a function of its surface. Express the surface as a function of the volume.

7. A right triangle has a hypotenuse 10 units long. From the vertex of the right angle the altitude and median are drawn to the hypotenuse. Express the area of the triangle as a function of the segment of the hypotenuse between the median and the altitude. *Ans.* $A = 5\sqrt{25 - x^2}$.

8. The velocity of a falling body varies as the square root of the distance it has fallen. If the velocity is 32 ft. per second when it has fallen 16 ft., express the distance as a function of the velocity.

Write the inverse of each of the following functions. (Nos. 9–16.)

9. $y = 1 - x^2$. *Ans.* $x = \pm\sqrt{1 - y}$.

10. $x = 2 - 3y + 2y^2$.

11. $y = 3\sin(x + \pi/2)$. *Ans.* $x = \cos^{-1}(y/3)$.

12. $x = 2\cos^{-1}(2y/3)$.

13. $y = \sin x \cos x$. *Ans.* $x = (1/2)\sin^{-1} 2y$.

14. $y = e^{2x}$.

15. $y = \log \sqrt{x^2 + a^2}$. *Ans.* $x = \pm \sqrt{e^{2y} - a^2}$.

16. $x = (1/2)\,(e^y + e^{-y})$.

Express explicitly each variable as a function of the other in the following cases. (Nos. 17–22.)

17. $x^{1/2} + y^{1/2} = 4$. *Ans.* $y = (4 - x^{1/2})^2$, $x = (4 - y^{1/2})^2$.

18. $x^{2/3} - y^{2/3} = 8$.

19. $x^2 + 2\,xy + 4\,y^2 = 5$.
Ans. $y = (1/4)(-x \pm \sqrt{20 - 3\,x^2})$, $x = -y \pm \sqrt{5 - 3\,y^2}$.

20. $3 \sin xy = 2$.

21. $2 \log x - 3 \log y = 4$. *Ans.* $y = (x/e^2)^{2/3}$, $x = \pm\, e^2 y \sqrt{y}$.

22. $2 \log xy = 5$.

51. Limits. An idea of fundamental importance in the calculus is that of the limiting value of a function of a variable.

Let $f(x)$ be any function of the variable x. Then, as x approaches any constant a, if the corresponding values of $f(x)$ approach a definite constant l in such manner that the numerical value of the difference $l - f(x)$ becomes and remains less than any preassigned positive number, however small, then l is said to be the limit of $f(x)$.

The notation

$$\lim_{x \to a} f(x) = l$$

is read " the limit of $f(x)$, as x approaches a, is l." In other words, this statement means that $f(x)$ is as near as we like to l if x is near enough to a.

The student is already familiar with illustrations of a function approaching a constant as a limit. Thus the sum of the first n terms in the geometric progression

$$S_n = 1 + 1/2 + 1/4 + 1/8 + \cdots + 1/2^{n-1} \tag{1}$$

is evidently a function of n, the number of terms involved. The limit which S_n approaches as n increases is 2, and the expression

$$\lim_{n \to \infty} S_n = 2$$

is read " the limit of S_n as n increases without limit is 2."

Again, the area C of a circle is defined as the limit approached by the area A of a regular inscribed or circumscribed polygon as the number, n, of its sides increases without limit, or

$$\lim_{n \to \infty} A = C.$$

A function may or may not attain its limit. In the series (1), the addition of successive terms will never add up to 2; nevertheless the limit exists.

52. Theorems on Limits. Certain theorems on limits are stated without proof. In each one the existence of the limit is implied.

THEOREM I. *The limit of an algebraic sum of any finite number of variables is equal to the same algebraic sum of their respective limits.*

THEOREM II. *The limit of the product of any finite number of variables is equal to the product of their respective limits.*

THEOREM III. *The limit of the quotient of two variables is equal to the quotient of their respective limits, provided the limit of the denominator is not zero.*

THEOREM IV. *If $f \leqq g \leqq h$ and if $\lim f = \lim h$, then we have $\lim g = \lim h$.*

53. Infinitesimals. *An **infinitesimal** is a variable whose limit is zero.* From the definition of the limit of a function it follows that the difference between a function and its limit is an infinitesimal. Hence if

$$\lim_{x \to a} f(x) = l$$

then $l - f(x)$ is an infinitesimal whenever $a - x$ is an infinitesimal. As other examples, we may state that if v is an infinitesimal, so are also kv (where k is any constant), $\sin v$, and $(1 - \cos v)$.

54. Continuous Functions. *A function $f(x)$ is said to be **continuous for $x = a$** if*

$$\lim_{x \to a} f(x) = f(a).$$

*A function is **continuous in the interval $x = x_1$ to $x = x_2$** if it is continuous for all values of x in this interval.*

*A function is said to be **discontinuous for $x = a$** if the condition for continuity is not satisfied.* The only functions which we shall consider are those which are in general continuous, but which may have a discontinuity for some value or values of the variable. As examples of discontinuity we mention the following:

(*a*) When the function $f(x)$ increases without limit or decreases without limit as x approaches a, that is, when

$$\lim_{x \to a} f(x) = \infty.$$

Thus the function $y = 1/x^2$ (Fig. 75) is continuous for every value of x excepting $x = 0$, for which it is not defined.

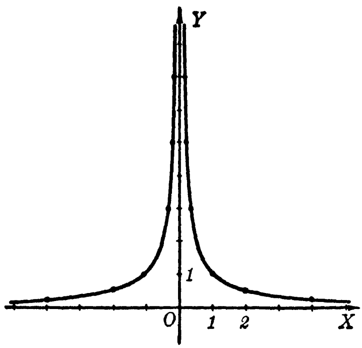


FIG. 75

(*b*) The function $f(x)$ may approach different values as a limit according as the variable x approaches a from a value greater than a, or less than a. Thus the function $y = 1/(1 - x)$ decreases indefinitely as x approaches 1 from a greater value, but increases indefinitely if x approaches 1 from a smaller value. This function is not defined for $x = 1$ but is continuous for all other values. (Fig. 76.)

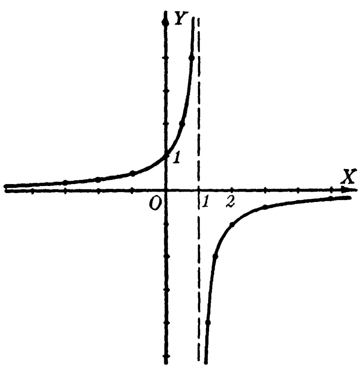


FIG. 76

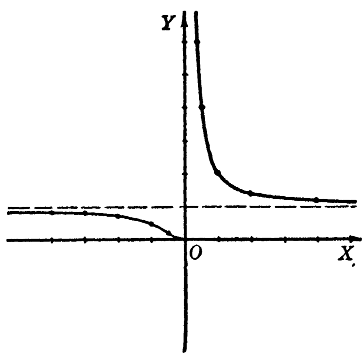


FIG. 77

(*c*) Another illustration is shown in the graph of

$$y = 2^{1/x}.$$

As x approaches 0 from positive values y increases without limit but as x approaches 0 from negative values y approaches the limit 0. This graph possesses what is known as an ***end point*** at the origin. The function is not defined for $x = 0$. (Fig 77.)

55. Evaluation of Limits. It is necessary to know the values of some important limits used in the calculus. Among these are:

$$(a) \qquad \lim_{\theta \to 0} \left[\frac{\sin \theta}{\theta} \right],$$

if θ is measured in radians.

In Fig. 78, OAB is a circular sector with AC a tangent at A, and central angle θ. We have

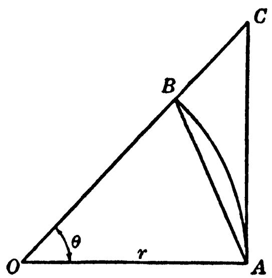


FIG. 78

$$\triangle\, OAB < \text{sector } OAB < \triangle\, OAC.$$

That is,

$$\frac{1}{2} r^2 \sin \theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \tan \theta$$

or, on dividing by $(r^2/2) \sin \theta$,

$$1 < \frac{\theta}{\sin \theta} < \sec \theta.$$

But $\lim\limits_{\theta \to 0} [\sec \theta] = 1$; therefore, by Theorem IV, § 52,

$$\lim_{\theta \to 0} \left[\frac{\theta}{\sin \theta} \right] = 1.$$

Hence the reciprocal ratio will have the limit 1, or

$$(1) \qquad \lim_{\theta \to 0} \left[\frac{\sin \theta}{\theta} \right] = 1.$$

$$(b) \qquad \lim_{\theta \to 0} \left[\frac{1 - \cos \theta}{\theta} \right].$$

Since

$$\frac{1 - \cos \theta}{\sin \theta} = \tan \frac{\theta}{2},$$

we have

$$\frac{1 - \cos \theta}{\theta} = \frac{\sin \theta}{\theta} \cdot \tan \frac{\theta}{2}.$$

Hence, by Theorem II, § 52,

$$\lim_{\theta \to 0} \left[\frac{1 - \cos \theta}{\theta} \right] = \lim_{\theta \to 0} \left[\frac{\sin \theta}{\theta} \right] \cdot \lim_{\theta \to 0} \left[\tan \frac{\theta}{2} \right] = 1 \cdot 0,$$

or

$$\lim_{\theta \to 0} \left[\frac{1 - \cos \theta}{\theta} \right] = 0. \tag{2}$$

(c) $\lim_{x \to 0} (1 + x)^{1/x}$. The proof that $\lim_{x \to 0} (1 + x)^{1/x}$ exists is beyond the scope of this book, but it is given in advanced courses* and will be assumed when needed.

Since this limit does exist, we can approximate its value by the following logarithmic calculation.

Let

$$y = (1 + x)^{1/x},$$

then

$$\log_{10} y = \log_{10} (1 + x)^{1/x} = (1/x) \log_{10} (1 + x).$$

By referring to a more extensive table of logarithms (an 8- or 10-place table), we obtain the values below.

x	$(1/x) \log_{10} (1 + x)$	$y = (1 + x)^{1/x}$
10^{-1}	0.41393	2.5938
10^{-2}	0 43214	2.7048
10^{-3}	0.43408	2.7169
10^{-4}	0 43427	2.7181
-10^{-1}	0 45757	2.8679
-10^{-2}	0.43648	2 7320
-10^{-3}	0.43451	2 7196
-10^{-4}	0.43432	2.7184

It is known that, as x approaches zero either from positive or negative values, $(1 + x)^{1/x}$ approaches a definite limit known as e, which is one of the most important constants in mathematics. To eight significant figures, it is

$$\lim_{x \to 0} (1 + x)^{1/x} = e = 2.7182818,$$

and to the same number of significant figures,

$$\log_{10} e = 0.43429448.$$

* Pierpont, *The Theory of Functions of Real Variables*, Vol. 1, §§ 306–308.

PROBLEMS

1. Discuss $e^{1/x}$ between $x = -1$ and $x = 1$.
2. Discuss $e^{1/(1-x)}$ between $x = -1$ and $x = 3$.
3. Discuss and draw the graph of $y = 1/(x-2)$ from $x = -3$ to $x = 4$.
4. Discuss and draw the graph of $y = 2/(x-1)^2$ from $x = -3$ to $x = 4$.
5. Find the nature of the function $\sin(1/x)$ as x approaches zero.
6. Draw a careful graph of $y = (1+x)^{1/x}$ from $x = -0.9$ to $x = 4$.
7. Evaluate (a) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$, (b) $\lim\limits_{\theta\to 0} \dfrac{\tan\theta}{\sin\theta}$. *Ans.* (a) 4; (b) 1.
8. Evaluate (a) $\lim\limits_{\theta\to 0} \dfrac{\sin 3\theta}{\sin\theta}$, (b) $\lim\limits_{\theta\to\pi/4} \dfrac{1+\cos 4\theta}{\cos 2\theta}$.
9. Evaluate (a) $\lim\limits_{\theta\to 0} \dfrac{\sin^3\theta}{\sin^3 2\theta}$, (b) $\lim\limits_{\theta\to 0} \dfrac{\sec^2\theta - 1}{2\sin^2\theta}$.

Ans. (a) 1/8; (b) 1/2.

56. Increments. If a variable changes from one value to another, the difference of the two values, obtained by subtracting the first value from the second, is called the ***increment of the variable.*** Thus if x changes from x_1 to x_2, the increment of x is $x_2 - x_1$. This increment is represented by the symbol Δx (read "delta x") so that $\Delta x = x_2 - x_1$. When the variable x is given an increment in a discussion or a problem, it is customary to assume the values x and $x + \Delta x$ rather than x_1 and x_2, respectively.

Now consider any function of x, such as $f(x) = x^2 - 6x + 7$. It is evident that a change in the value of the variable x will produce in general a change in the function. Thus, when $x = 2$, $f(2) = -1$, and if $\Delta x = 4$, that is, $x + \Delta x = 6$, then $f(6) = 7$. As x changes to $x + \Delta x$ the function changes from $f(x)$ to $f(x + \Delta x)$; the difference $f(x + \Delta x) - f(x)$ is called $\Delta f(x)$, ***the increment of the function.*** Or, if we call the function y, an increment Δx assigned to the variable produces a corresponding increment Δy in the function so that $y + \Delta y = f(x + \Delta x)$ and $\Delta y = f(x + \Delta x) - f(x)$. Hence the increment of the function, Δy, is expressed in terms of both the variables x and Δx.

EXAMPLE

Given the function $y = x^2 - 6x + 7$. Calculate the increment of the function.

Solution. First assign the variable x an increment Δx. Then

$$y + \Delta y = f(x + \Delta x) = (x + \Delta x)^2 - 6(x + \Delta x) + 7.$$

But

$$y \qquad = f(x) \qquad = x^2 \qquad - 6x \qquad + 7.$$

Subtracting, we find

$$\Delta y = f(x + \Delta x) - f(x) = 2\,x \cdot \Delta x + \overline{\Delta x}^2 - 6 \cdot \Delta x,$$

which is a function of x and Δx.

57. Average Rate of Change. Consider the function $y = f(x)$. A change of Δx in the variable produces a corresponding change of Δy in the function, *the ratio $\Delta y/\Delta x$ is defined as* ***the average rate of change of the function with respect to the variable*** *in the given interval x to $x + \Delta x$.*

Let the graph of the function be drawn and mark on it any general point $P(x, y)$. Give x an increment Δx and mark the point Q on the graph whose abscissa is $(x + \Delta x)$. Its ordinate is $(y + \Delta y)$. Figure 79 is the graph of the function

$$y = x^2 - 6\,x + 7.$$

The points P and Q of the graph are taken at $(2, -1)$ and $(6, 7)$, respectively, which correspond to the values of x and $x + \Delta x$ arbitrarily selected in the preceding article. Here a change in the variable from 2 to 6, making $\Delta x = 4$, produced a change in the function from -1 to 7, making $\Delta y = 8$. Hence in this interval the average rate of change of the function with respect to the variable, that is, $\Delta y/\Delta x$, is 2.*

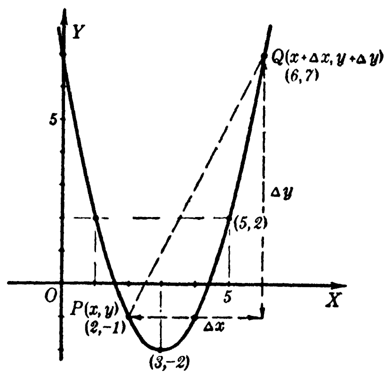


FIG. 79

From the graph above it is evident that $\Delta y/\Delta x$ is the difference of the ordinates of P and Q divided by the difference of the abscissas taken in the same order and is, therefore, the slope of the secant line PQ. The same statement would hold true for the graph of any other function. Hence we have the following important relation.

The ***average rate of change of a function with respect to a variable*** *in a given interval is equal to the slope of the secant line*

* This does not mean that the function is changing twice as fast as the variable throughout the interval. An examination of the graph reveals that in the first part of this interval, as the variable changes from 2 to 3, the function is actually decreasing; while in the latter part of the interval, as the variable changes from 5 to 6, the function increases from 2 to 7, which is five times the corresponding change in the variable.

joining the two points on the graph of the function corresponding to the extremities of the interval.

PROBLEMS

Express Δy as a function of x and Δx and find its value corresponding to the values given in each of the following cases. (Nos. 1–8.)

1. $y = 2x^2$, $x = 2$, $\Delta x = 0.5$. *Ans.* $4x \cdot \Delta x + 2\overline{\Delta x}^2$, 4.5.
2. $y = x - 3x^2$, $x = -1$, $\Delta x = 1$.
3. $y = x^3 - x + 4$, $x = 2$, $\Delta x = 0.25$.
   Ans. $3x^2 \cdot \Delta x + 3x \cdot \overline{\Delta x}^2 + \overline{\Delta x}^3 - \Delta x$, $3\frac{9}{64}$.
4. $y = 5x - x^3$.
5. $y = x + 1 + 1/x^2$, $x = 3$, $\Delta x = 2$.
   Ans. $\Delta x - (2x \cdot \Delta x + \overline{\Delta x}^2)/x^2(x + \Delta x)^2$, $434/225$.
6. $y = x - 1/x$, $x = 4$, $\Delta x = 2$.
7. $y = x^2 - 2/x^2$, $x = -3$, $\Delta x = 2$.
   Ans. $2x \cdot \Delta x + \overline{\Delta x}^2 + (4x \cdot \Delta x + 2\overline{\Delta x}^2)/x^2(x + \Delta x)^2$, $-88/9$.
8. $y = x^3 - 2/x$, $x = 3$, $\Delta x = 0.01$

Find the average rate of change of each of the following functions and evaluate for the given interval. (Nos. 9–18.)

9. $s = (1 + t)/(1 - t)$ for Δt. *Ans.* $2/[(1 - t)(1 - t - \Delta t)]$.
10. $s = 100t - 16t^2$, $t = 3$, $\Delta t = 2$.
11. $f(x) = 2 - x/(x - 1)$, $x = 0$, $\Delta x = 0.5$. *Ans.* 2.
12. $f(y) = 2y/(2 - 3y^2)$, $y = 2$, $\Delta y = 1$.
13. $f(t) = t^2 - t + 1/t$, $t = 2$, $\Delta t = -0.5$. *Ans.* $2\frac{2}{3}$.
14. $f(x) = x/(x^2 - 1) + 3$, $x = 2.5$, $\Delta x = -0.5$.
15. $y = \sqrt{2x + 1}$, $x = 4$, $\Delta x = 0.3$. *Ans.* 0.328.
16. $y = 1/\sqrt{3 + x}$, $x = 3$, $\Delta x = 2$.
17. $f(y) = 1/y - y^3 - 4$, $y = 3$, $\Delta y = 1$. *Ans.* $-37\frac{1}{12}$.
18. $f(x) = \sqrt{4 - x^2}$, $x = 1$, $\Delta x = -0.25$.

58. The Derivative. Many important properties of a function of a variable are found with the aid of a related function called *the derivative of the function with respect to the variable.* Let the given function be $y = f(x)$. The derivative is obtained as follows:

First, assign to the variable x an increment Δx and calculate Δy, the corresponding increment of the function.

Next, divide Δy by Δx and evaluate the limit of this quotient as Δx approaches zero. That is, find

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

This limit is ***the derivative of y with respect to x.***

We have seen (§ 56) that Δy is a function of x and Δx. The same is true of the quotient $\Delta y/\Delta x$, but the limit of this quotient as Δx approaches zero, namely, the derivative, is a function of x alone.

The symbol most frequently used to represent the derivative with respect to the independent variable x is d/dx. Thus the derivative of y with respect to x is dy/dx or $(d/dx)f(x)$. The primed symbols y' or $f'(x)$ are also used. Some texts use the symbols D_x and $D_x y$ in place of d/dx and dy/dx, respectively. It is important to remember that whatever symbol is used it represents the result obtained by performing the operations above on the given function. Thus,

$$(1) \qquad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Similarly, if u is a function of t, then $du/dt = \lim\limits_{\Delta t \to 0} (\Delta u/\Delta t)$, or, if $s = f(v)$, then s' or $f'(v) = \lim\limits_{\Delta v \to 0} (\Delta s/\Delta v)$.

EXAMPLES

1. Find dy/dx if $y = x^2 - 3x$.

Solution. Give x an increment Δx and calculate Δy. Thus

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 - 3(x + \Delta x), \\ y &= x^2 \qquad\qquad - 3x. \\ \hline \Delta y &= 2x \cdot \Delta x + \overline{\Delta x}^2 - 3 \cdot \Delta x. \end{aligned}$$

Dividing both sides by Δx, and calculating the $\lim\limits_{\Delta x \to 0} (\Delta y/\Delta x)$, we find

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x - 3,$$

whence

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 3) = 2x - 3,$$

or

$$\frac{dy}{dx} = 2x - 3.$$

2. Given $f(t) = (2/t) + 5$, find $f'(t)$.

SOLUTION. Assign the independent variable t an increment Δt, then

$$f(t + \Delta t) = \frac{2}{t + \Delta t} + 5,$$

$$\Delta f(t) = f(t + \Delta t) - f(t) = \frac{2}{t + \Delta t} - \frac{2}{t} = \frac{-2 \cdot \Delta t}{t(t + \Delta t)},$$

whence

$$\frac{\Delta f(t)}{\Delta t} = -\frac{2}{t(t + \Delta t)},$$

and

$$f'(t) = \lim_{\Delta t \to 0} \frac{\Delta f(t)}{\Delta t} = \lim_{\Delta t \to 0} \left[-\frac{2}{t(t + \Delta t)} \right] = -\frac{2}{t^2}.$$

3. Find d/dx of $ax^2 + 2\,bx + c$.

SOLUTION. Let $y = ax^2 + 2\,bx + c$. Then we have

$$y + \Delta y = a(x + \Delta x)^2 + 2\,b(x + \Delta x) + c,$$

hence

$$\Delta y = 2\,ax \cdot \Delta x + a \cdot \overline{\Delta x}^2 + 2\,b \cdot \Delta x,$$

and

$$\frac{\Delta y}{\Delta x} = 2\,ax + a \cdot \Delta x + 2\,b.$$

Therefore

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2\,ax + a \cdot \Delta x + 2\,b) = 2\,ax + 2\,b;$$

that is,

$$\frac{d}{dx}(ax^2 + 2\,bx + c) = 2(ax + b).$$

The student should now be able to comprehend the following definition.

DEFINITION OF A DERIVATIVE. *Given a continuous function of a variable, if the increment of the function is divided by the increment of the variable, the limit of the quotient, as the increment of the variable approaches zero, is called* ***the derivative of the function with respect to the variable.***

All functions which we shall consider will be ***differentiable,*** that is, the derivative of the function is in general another continuous function which may become discontinuous only for particular values of the variable. For any value of x for which the limit of $\Delta y/\Delta x$ exists, the derivative is said to *exist.*

PROBLEMS

Find the derivative of each of the following functions. (Nos. 1–17.)

1. $y = x^2 - x + 4$. *Ans.* $2\,x - 1$.

2. $y = x^3 - 5\,x$.

3. $y = x^3 + 4x^2 - 5x + 7.$ *Ans.* $3x^2 + 8x - 5.$

4. $y = (x^2 - x - 1)^2.$

5. $y = 1/(x - 1).$ *Ans.* $-1/(x - 1)^2.$

6. $f(x) = 1/(3 - 2x).$

7. $f(t) = 32 + 100t - 16t^2.$ *Ans.* $100 - 32t.$

8. $f(y) = 2 + y/(y - 1).$

9. $f(y) = y^2 - y - 4/y.$ *Ans.* $2y - 1 + 4/y^2.$

10. $f(x) = x^2 - 1/x^2$ at $x = 1.$

11. $f(x) = 2x/(2 - 3x).$ *Ans.* $4/(2 - 3x)^2.$

12. $f(t) = t^2 - (t - 2)/(t + 2).$

13. $f(s) = 2s/(4 - 3s^2).$ *Ans.* $(8 + 6s^2)/(4 - 3s^2)^2.$

14. $f(x) = \sqrt{1 - x}.$ (HINT: Rationalize numerator of $\Delta y/\Delta x$.)

15. $f(x) = \sqrt{x + 1} - \sqrt{x}.$ *Ans.* $1/(2\sqrt{x + 1}) - 1/(2\sqrt{x}).$

16. $y = ax/(x - a).$

17. $f(u) = a/(a^2 - u^2).$ *Ans.* $2au/(a^2 - u^2)^2.$

18. If y has a derivative with respect to x for a given value of x, what condition must be satisfied by Δy as $\Delta x \to 0$?

19. Find $f'(t)$ if $f(t) = (t - 3)^{-1/2}.$
Ans. $-(1/2)(t - 3)^{-3/2}.$

20. Find the derivative of any of the functions given in the problems on p. 80.

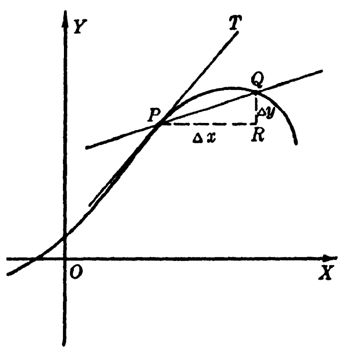


FIG. 80

59. Geometric Interpretation of the Derivative.

59. Geometric Interpretation of the Derivative. Let us assume Fig. 80 to be the graph of $y = f(x)$ with $P(x, y)$ any point on it. Give x an increment Δx, then $\Delta y/\Delta x$ is the tangent of $\angle RPQ$, or the slope of the secant line joining $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ (see § 57). Let Δx approach zero, then, since $f(x)$ is a continuous function, Δy approaches zero. That is, the point Q moves along the graph and approaches P as a limit. Hence if $\Delta y/\Delta x$ has a limit, it is the slope of the limiting position of the secant line. But, by definition, we have

The ***tangent to any curve*** *at a point P is the limiting position of the secant joining P and another point Q on the curve as Q approaches P.*

Therefore, $\lim\limits_{\Delta x \to 0} (\Delta y/\Delta x)$ is the slope of the tangent line to the graph of $y = f(x)$ at the point $P(x, y)$. Or, we can say:

*The **numerical value of the derivative** dy/dx for any given value x_1 assigned to the variable is **the slope of the tangent line at the point** on the graph of $y = f(x)$ **whose abscissa is x_1**.*

EXAMPLE

Draw the graph of $3y = x^3 - 2x^2 - 4x$ by making a table of values and showing the slope at each point marked.

SOLUTION. Finding the derivative, we obtain

$$y' = x^2 - \frac{4}{3}x - \frac{4}{3}.$$

Make a table of values of x, y, and y'. Through each point (x, y) draw an arrow in the direction indicated by y', the slope of the tangent.

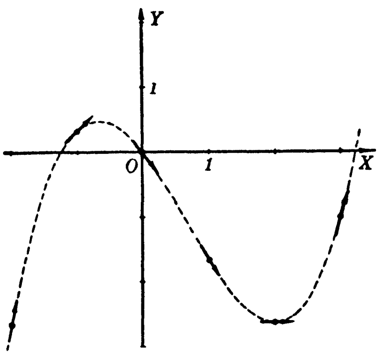


FIG. 81

x	y	y'
-2	$-8/3$	$16/3$
-1	$1/3$	1
0	0	$-4/3$
1	$-5/3$	$-5/3$
2	$-8/3$	0
3	-1	$11/3$

60. Physical Interpretation of the Derivative. Let two physical quantities be connected by a functional relation. Calling them u and v we have, $u = f(v)$. Then any change Δv in the variable v produces a corresponding change Δu in u. *The ratio of these increments, $\Delta u/\Delta v$ is the **average rate of change of u with respect to v** in the interval Δv.* Now let Δv approach zero; then Δu approaches the limit zero, since u is assumed to be a continuous function of v. Hence if $\Delta u/\Delta v$ has a limit, it is defined to be the rate of change of u with respect to v at the beginning of the interval. That is:

*The $\lim\limits_{\Delta v \to 0} (\Delta u/\Delta v)$, or du/dv is **the exact rate of change of the function u with respect to the variable v** and is measured in units of u per unit v.*

Thus the distance s, in feet, of an object falling by the influence of gravity is a function of the time t, in seconds. An increment of time Δt changes the distance by Δs. Then $\Delta s/\Delta t$ is the average rate of change of s per unit t in the interval. This we call the

average velocity during the interval and is measured in feet per second. The value of ds/dt for any given t is the ***exact velocity*** at that instant in feet per second.

Also the temperature T at which water boils is a function of h, the altitude. If T is measured in degrees C, and h in meters, then dT/dh is the rate of change of degrees C per unit h, that is, degrees per meter.

Again, the pressure p and the volume v of a gas in a container are connected by a functional relation. If p is measured in pounds per unit area and v in cubic inches, then dp/dv for any value of v is the rate of change of p with respect to v, which is measured in (lbs. per unit area) per cu. in.

EXAMPLE

If a body falls from rest under the influence of gravity, the relation between the velocity v in feet per second, and the distance fallen s in feet, is approximately $v = 8\sqrt{s}$. (*a*) At what rate is v changing with respect to s when s is 4 feet? 36 feet? (*b*) For what value of s are v and s changing at the same rate?

SOLUTION. To find dv/ds, give s an increment Δs and calculate Δv. Then

$$v + \Delta v = 8\sqrt{s + \Delta s},$$
$$\Delta v = 8(\sqrt{s + \Delta s} - \sqrt{s}),$$
$$\Delta v/\Delta s = 8(\sqrt{s + \Delta s} - \sqrt{s})/\Delta s.$$

Rationalizing the numerator, we have

$$\Delta v/\Delta s = 8\,\Delta s/\Delta s(\sqrt{s + \Delta s} + \sqrt{s})$$
$$= 8/(\sqrt{s + \Delta s} + \sqrt{s}),$$

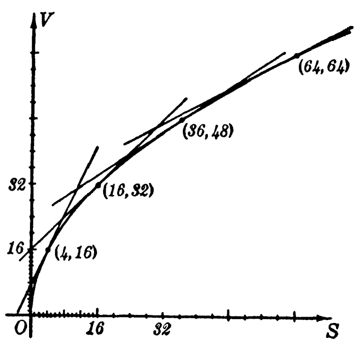


FIG. 82

whence

$$\frac{dv}{ds} = \lim_{\Delta s \to 0} \frac{\Delta v}{\Delta s} = \lim_{\Delta s \to 0} \frac{8}{\sqrt{s + \Delta s} + \sqrt{s}} = \frac{4}{\sqrt{s}}.$$

(*a*) When $s = 4$ ft., $dv/ds = 2$, that is, v is changing at the rate of 2 units of v per unit change in s. When $s = 36$ ft., $dv/ds = 2/3$, and therefore v is changing at the rate of 2/3 units v per unit s, or 2/3 (ft./sec.) per ft.

(*b*) The velocity v is changing at the same rate as s when $dv/ds = 1$. That is

$$\frac{4}{\sqrt{s}} = 1, \qquad s = 16 \text{ ft.}$$

This does not mean that v and s are the same, for when $s = 16$ ft., v is 32 ft. per sec., but v is then changing at the same rate as s is changing.

Draw the graph of $v = 8\sqrt{s}$, taking s as abscissa and v as ordinate. By

using dv/ds, we find the slopes of the tangents at $s = 4$ and $s = 36$ to be 2 and 2/3, respectively. Also the slope of the tangent is 1 at the point where $s = 16$.

We observe that s and v are numerically the same in their respective units when $s = 64$, but then $dv/ds = 1/2$; that is, v is then changing 1/2 as fast as s is changing.

PROBLEMS

Form a table for x, y, and dy/dx for each of the following equations and plot points showing the direction of the curve at each point. (Nos. 1–10.)

1. $y = x^2 + 4x$.

2. $x^2 = 6 - y$.

3. $2xy = 9$.

4. $x = 4y - y^2$.

5. $3x = (y - 1)^2$.

6. $(y - 3)(x - 2) = 6$.

7. $y = x^3$ from $x = -2$ to $x = 2$.

8. $y = 3x - x^3$.

9. $xy - 5x^2 = 4$.

10. $3y = x^4 - 4x$.

11. Find the rate of change of the area of a sector of a circle of radius a units with respect to the angle. *Ans.* $a^2/2$ sq. units per unit of angle.

12. The distance a body falls from rest under the force of gravity is $s = 16t^2$. Find its velocity at any time. Find its acceleration. What is its velocity and the distance fallen after 3 seconds? What is its velocity after it has fallen 48 ft.?

13. The law connecting the pressure and the volume of a fixed quantity of gas at constant temperature is $pv = c$. What is the rate of change of p with respect to v when v is 4 cubic units? What is the rate of change of v with respect to p when v is 4 cubic units? How are these two rates related?

Ans. $-c/16$ units p per unit v; $-16/c$ units v per unit p; reciprocals.

14. A ball thrown vertically upward has its distance from the starting point given by $s = 100t - 16t^2$. When does it stop rising? What is its velocity at the end of 4 seconds?

15. Find the values of x for which the tangents to $y = 3x^2$ and $y = x^3$ are parallel. *Ans.* 0, 2.

16. Find the direction a particle is moving at the point determined by $x = 2$ if it follows the graph of $y = 2x - x^2$.

17. Find the rate of change of the volume of a sphere with respect to its radius. Evaluate the rate when $r = 3$ units. *Ans.* 36π cubic units per unit r.

18. Find the rate of change of the volume of a sphere with respect to its diameter. Evaluate for $r = 2$ units.

19. Set up the volume of a solid made up of a right circular cylinder with a hemisphere on each end. If the length of the cylinder is twice the radius of an end, find the rate of change of the volume of the solid with respect to the radius of one end. *Ans.* $10\pi r^2$ cubic units per unit r.

20. In Problem 19, find the rate of change of the solid with respect to its total length.

ADDITIONAL PROBLEMS

Find the average and exact rate of change of the following functions. (Nos. 1–8.)

1. $3x - x^2$. *Ans.* $3 - 2x - \Delta x, \quad 3 - 2x$.

2. $4 - 2y - y^2$.

3. $1/u - u^2$. *Ans.* $-1/u(u + \Delta u) - 2u - \Delta u, \quad -1/u^2 - 2u$.

4. $u/(u - 1)$.

5. $v^3 - 1/v$. *Ans.* $3v^2 + 3v \cdot \Delta v + \overline{\Delta v}^2 + 1/v(v + \Delta v), \quad 3v^2 + 1/v^2$.

6. $2x^2 - 3/x$.

7. $2/\sqrt{x - 3}$.
Ans. $-2/\sqrt{x - 3}\sqrt{x + \Delta x - 3}(\sqrt{x + 3} + \sqrt{x + \Delta x - 3})$,
$-(x - 3)^{-3/2}$.

8. $\sqrt{x + 1/x}$.

Draw the graph of each of the following equations with the help of the derivative. (Nos. 9–14.)

9. $y = 3x + x^2$.

10. $xy = -8$.

11. $2y = 3x - x^3$.

12. $3x = y^2 + 2y - 5$.

13. $6y = 4x - x^4$.

14. $xy - 2x^2 - 4 = 0$.

15. Find points on $xy - 5x^2 = 4$ where the slope of the tangent is 1.
Ans. $(1, 9), (-1, -9)$.

Express each of the following quantities as a function of the suggested variable. Find the rate of change of each function with respect to its variable. (Nos. 16–24.)

16. The area of a circle in terms of its circumference.

17. The volume of a box with square base as a function of its altitude if h is 3 times a side of its base. *Ans.* $h^3/9$, $h^2/3$.

18. The surface of a box with square base as a function of a side of the base if its volume is constant.

19. The total surface of a circular cylinder as a function of the radius of one end if its volume is constant. *Ans.* $2\pi r^2 + 2V/r$, $4\pi r - 2V/r^2$.

20. The volume of a box made by cutting squares from the corners of a rectangular sheet 12″ by 6″ as a function of the side of the squares.

21. The total surface of a cone as a function of its altitude if $r = 2h$.
Ans. $2\pi h^2(2 + \sqrt{5})$, $4\pi h(2 + \sqrt{5})$.

22. The volume of a cone as a function of its altitude if $r = (h/2)$.

23. The volume of a sphere as a function of the area of a great circle.
Ans. $(A/6\pi)\sqrt{\pi A}$, $(1/4\pi)\sqrt{\pi A}$.

24. The volume of a sphere in terms of its surface.

CHAPTER IV

DIFFERENTIATION OF ALGEBRAIC, LOGARITHMIC, AND EXPONENTIAL FUNCTIONS

61. Derivation of Formulas. The method of forming the derivative of a function, as explained in the last chapter, is perfectly general and can be applied to all differentiable functions. However, by differentiating a special type of function we obtain a formula which, when memorized, may be used to write down the derivative of any function belonging to that type.

A thorough knowledge of the formulas derived in the following articles is essential.

62. Derivative of a Constant. Let the function be a contant c. Call it y and write $y = c$. Then any increment Δx assigned to the independent variable x will not affect the function, since it is constant. Hence we have

$$y + \Delta y = c, \qquad \text{and} \qquad \Delta y = 0,$$

whence

$$\frac{\Delta y}{\Delta x} = 0,$$

and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 0.$$

Therefore

$$\text{(I)} \qquad \frac{dc}{dx} = 0.$$

The derivative of a constant with respect to any variable is ***zero.***

This result is evident if we consider the graph of $y = c$, which is a straight line parallel to the x axis. For any two points (x, c) and $(x + \Delta x, c)$ on the line, the rate of change of y with respect to x is zero. In other words, the slope of the graph is always zero.

63. Derivative of the Independent Variable. Let the function be x. Then an increment Δx will produce the same increment in the function y. That is, $\Delta y = \Delta x$.

Then

$$\frac{\Delta y}{\Delta x} = 1,$$

and

$$\frac{dy}{dx} = 1,$$

or

(II)
$$\frac{dx}{dx} = 1.$$

The derivative of a variable with respect to itself is ***unity.***
Is this result apparent from the graph of $y = x$? Explain.

64. Derivative of a Constant Times a Function. Given the function cu, where c is a constant and u is any differentiable function of x. Give x an increment Δx; this will change u to $u + \Delta u$, and the function $y = cu$ to

$$y + \Delta y = c(u + \Delta u),$$

then

$$\Delta y = c \cdot \Delta u,$$

$$\frac{\Delta y}{\Delta x} = c\frac{\Delta u}{\Delta x},$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} c\frac{\Delta u}{\Delta x} = c \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = c\frac{du}{dx},$$

since u is a differentiable function of x. Hence

(III)
$$\frac{d(cu)}{dx} = c\frac{du}{dx}.$$

The derivative of a constant times a function is equal to the ***constant times the derivative of the function.***

65. Derivative of an Algebraic Sum of Functions. Let u, v, and w be any differentiable functions of x, and consider the function

$$y = u + v - w.$$

Give x an increment Δx. This will cause u, v, and w each to assume a corresponding increment. Then

$$y + \Delta y = u + \Delta u + v + \Delta v - (w + \Delta w),$$

$$\Delta y = \Delta u + \Delta v - \Delta w,$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} - \frac{\Delta w}{\Delta x},$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} - \frac{\Delta w}{\Delta x}\right] = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx},$$

by Theorem I, § 52. Therefore

$$\text{(IV)} \qquad \frac{d(u + v - w)}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}.$$

The derivative of an algebraic sum of functions is the corresponding ***algebraic sum of the derivatives of the functions.*** A finite number of functions is assumed.

66. Derivative of a Power of a Function. Let $y = u^n$, u being any differentiable function of x. Give x an increment Δx, then

$$y + \Delta y = (u + \Delta u)^n.$$

Assuming n to be *a positive integer* and expanding by the binomial theorem, we have

$$y + \Delta y = u^n + nu^{n-1}\cdot\Delta u + \frac{n(n-1)}{1\cdot 2}u^{n-2}\cdot\overline{\Delta u}^2 + \cdots + \overline{\Delta u}^n,$$

$$\Delta y = nu^{n-1}\cdot\Delta u + \frac{n(n-1)}{1\cdot 2}u^{n-2}\cdot\overline{\Delta u}^2 + \cdots + \overline{\Delta u}^n,$$

$$\frac{\Delta y}{\Delta x} = nu^{n-1}\frac{\Delta u}{\Delta x} + \frac{n(n-1)}{1\cdot 2}u^{n-2}\frac{\Delta u}{\Delta x}\cdot\Delta u + \cdots + \frac{\Delta u}{\Delta x}\cdot\overline{\Delta u}^{n-1}.$$

Taking the limit of both sides as Δx approaches zero and keeping in mind that $\lim\limits_{\Delta x \to 0} \Delta u = 0$, since u is a continuous function of x, and that du/dx exists, then

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx},$$

or*

$$\text{(V)} \qquad \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$

Since Formula **V** is known to be perfectly general, we shall use it for all values of n.

67. Derivative of the Product of Two Functions. Let u and v be any differentiable functions of x. We wish to find dy/dx when

$$y = uv.$$

* Formula **V** is derived on the assumption that n is a positive integer. We shall prove (§ 70) that the formula is true for n any rational number, and we shall discuss the general case in § 73.

Assign to x an increment Δx. Then

$$y + \Delta y = (u + \Delta u)(v + \Delta v),$$

$$\Delta y = u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v,$$

and

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u \cdot \Delta v}{\Delta x}.$$

Then

$$\frac{dy}{dx} = u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta u \cdot \Delta v}{\Delta x}.$$

The last term may be written either $\lim\limits_{\Delta x \to 0} \Delta u \cdot \lim\limits_{\Delta x \to 0} (\Delta v/\Delta x)$, or $\lim\limits_{\Delta x \to 0} (\Delta u/\Delta x) \cdot \lim\limits_{\Delta x \to 0} \Delta v$, either of which is zero since du/dx and dv/dx exist and $\lim\limits_{\Delta x \to 0} \Delta u = \lim\limits_{\Delta x \to 0} \Delta v = 0$. Hence

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx},$$

or

$$\text{(VI)} \qquad \boldsymbol{\frac{d(u \cdot v)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}.$$

The derivative of the product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.

68. Derivative of the Quotient of Two Functions. Let u and v be any differentiable functions of x. We wish to find dy/dx where y is u/v, and v is not zero. Proceed in the usual way to assign an increment Δx to x. Then

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v},$$

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v \cdot \Delta u - u \cdot \Delta v}{v(v + \Delta v)},$$

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}.$$

Taking the limit of both sides as Δx approaches zero and remembering that du/dx and dv/dx exist and that $\lim\limits_{\Delta x \to 0} \Delta v = 0$, we find

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2},$$

or

$$\text{(VII)} \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

The derivative of the quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all divided by the square of the denominator.

SPECIAL CASES. (*a*) If $u = c$, $y = c/v$, and

$$\textbf{(VII a)} \qquad \frac{d}{dx}\left(\frac{c}{v}\right) = -\frac{c}{v^2}\cdot\frac{dv}{dx}.$$

This is the same result we would obtain by differentiating $y = cv^{-1}$ by Formula **V**. The student can now prove the validity of Formula **V** for n any ***negative integer,*** by differentiating the function $1/v^n$ by Formula **VII *a*.**

(*b*) If $v = c$, $y = u/c = (1/c)u$, and

$$\textbf{(VII b)} \qquad \frac{d}{dx}\left(\frac{u}{c}\right) = \frac{1}{c}\cdot\frac{du}{dx}.$$

This is just a repetition of Formula **III**, for if c is a constant so is $1/c$. This case should always be recognized as a constant $(1/c)$ times a function and differentiated accordingly.

EXAMPLES

1. Differentiate $x^3 - 3x^2/2 + 5x - 7$ with respect to x.

SOLUTION. Let $y = x^3 - 3x^2/2 + 5x - 7$, whence y is an algebraic sum of functions. Then $d(x^3)/dx = 3x^2$, by **V**; $d(3x^2/2)/dx = 3x$, by **III** and **V**; $d(5x)/dx = 5$, by **III** and **II**; and $d(7)/dx = 0$, by **I**. Hence

$$\frac{dy}{dx} = 3x^2 - 3x + 5.$$

2. Find dy/dx if $y = (x^3 - 3x^4)^5$.

SOLUTION. Here $y = u^5$ where $u = x^3 - 3x^4$. But, by **V**,

$$\frac{dy}{dx} = 5u^4\frac{du}{dx},$$

and, as in Example 1,

$$\frac{du}{dx} = \frac{d}{dx}(x^3 - 3x^4) = 3x^2 - 12x^3.$$

Substituting for u and du/dx their respective values, we have

$$\frac{dy}{dx} = 5(x^3 - 3x^4)^4\cdot(3x^2 - 12x^3) = 15x^{14}(1 - 3x)^4(1 - 4x).$$

3. Find dy/dx if $y = (x^2 - 2)(x - 3x^3)$.

SOLUTION. This function is a product $u \cdot v$ where u is $(x^2 - 2)$ and v is $(x - 3x^3)$. Then, by **VI**, we have

$$\begin{aligned}\frac{dy}{dx} &= (x^2 - 2)\frac{d}{dx}(x - 3x^3) + (x - 3x^3)\frac{d}{dx}(x^2 - 2)\\ &= (x^2 - 2)(1 - 9x^2) + (x - 3x^3)(2x)\\ &= -15x^4 + 21x^2 - 2.\end{aligned}$$

4. Differentiate $(x - 3x^2)/\sqrt{2x - 5x^3}$ with respect to x.

SOLUTION. This is a quotient u/v, and hence by **VII**, calling y the function, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sqrt{2x - 5x^3}(1 - 6x) - (x - 3x^2)\cdot(1/2)(2x - 5x^3)^{-1/2}\cdot(2 - 15x^2)}{2x - 5x^3}\\ &= \frac{2(2x - 5x^3)(1 - 6x) - (x - 3x^2)(2 - 15x^2)}{2(2x - 5x^3)^{3/2}}\\ &= \frac{2x - 18x^2 + 5x^3 + 15x^4}{2(2x - 5x^3)^{3/2}}.\end{aligned}$$

PROBLEMS

Differentiate each of the following functions with respect to its variable.

1. $3x^2 - 4x^3 - 7$. *Ans.* $6x - 12x^2$.

2. $3x^3 - 3/x^3 + x^3/3$.

3. $ax^2 - bx + c + dx^{-1}$. *Ans.* $2ax - b - dx^{-2}$.

4. $(3x - 1)^2(x - 1)^3$.

5. $(x^2 - 2x)^3$. *Ans.* $6(x^2 - 2x)^2(x - 1)$.

6. $(x^4 + 4)(4 - x)^2$.

7. $(a - x^2)/(a + x^2)$. *Ans.* $-4ax/(a + x^2)^2$.

8. $7/(y^3 + 8)$.

9. $\sqrt{1 + y} + \sqrt{1 - y}$. *Ans.* $(\sqrt{1 - y} - \sqrt{1 + y})/(2\sqrt{1 - y^2})$.

10. $(y^2 - 2)^{3/2}$.

11. $(x^2 + 4)^{3/2} - (4 - x)^{-3/2}$. *Ans.* $3x(x^2 + 4)^{1/2} - 3(4 - x)^{-5/2}/2$.

12. $(\theta + 1)\sqrt{\theta^2 - 1}$.

13. $(y^3 - 2)^4/(y^2 + 2)$. *Ans.* $2y(y^3 - 2)^3(5y^3 + 12y + 2)/(y^2 + 2)^2$.

14. $\sqrt{x/2} - \sqrt{2x} + \sqrt{1/2x} - \sqrt{2/x}$.

15. $(x^2 + 2x)/\sqrt{1 - x}$. *Ans.* $(4 + 2x - 3x^2)/2(1 - x)^{3/2}$.

16. $\sqrt[3]{(1 - x^2)(1 + 2x)}$.

17. $\sqrt[3]{x/(x^3 + 1)}$. *Ans.* $(1 - 2x^3)/3x^{2/3}(x^3 + 1)^{4/3}$.

18. $v/(1 - \sqrt{1 - v^2})$.

19. $u/(u^2 - \sqrt{u^2 - 4})$. *Ans.* $(4 - u^2\sqrt{u^2 - 4})/\sqrt{u^2 - 4}(u^2 - \sqrt{u^2 - 4})^2$.

20. $x\sqrt{x^2 - a^2} - a^2x/\sqrt{x^2 - a^2}$.

21. $[(a + bx^2)/(a - bx^2)]^{1/4}$. *Ans.* $(abx)/(a + bx^2)^{3/4}(a - bx^2)^{5/4}$.

22. $(a^2 - x^2)/\sqrt{a^2 + x^2}$.

23. $(\sqrt{x^2 + 1} + \sqrt{x^2 - 1})/(\sqrt{x^2 + 1} - \sqrt{x^2 - 1})$.

Ans. $2(x + x^3/\sqrt{x^4 - 1})$.

24. $\sqrt{at^2 - 2\,bt}/\sqrt{2\,ct - d}$ in three ways, using **V**, **VI**, and **VII**.

25. Suppose $y = au + bv^2$, where u and v are differentiable functions of x. What formulas are used when you obtain dy/dx?

26. Suppose $y = (au \cdot v^2)/w^{1/2}$, where u, v, and w are differentiable functions of t. What formulas are used in obtaining dy/dt?

69. Derivative of a Function of a Function. If y is a function of u and u in turn is a continuous function of x, then an increment Δx assigned to x will produce a corresponding increment Δu in u, and Δy in y. For any value of these increments, provided Δu is not zero,* we have

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x},$$

whence, taking the limit of both sides, we find

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}\right] = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x},$$

since Δu approaches the limit zero as Δx approaches zero. Then

$$\textbf{(VIII)} \qquad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

If y is a function of u and u is a function of x, the derivative of y with respect to x is the product of the derivatives of y with respect to u, and of u with respect to x.

To express y directly as a function of x, we must eliminate u between the two given functions.

From Formula **VIII** we have at once:

$$\textbf{(VIII a)} \qquad \frac{dy}{du} = \frac{\dfrac{dy}{dx}}{\dfrac{du}{dx}}.$$

* For a proof of **VIII** when $\Delta u = 0$, see Pierpont, *Theory of Functions of Real Variables*, Vol. I, p. 234.

70. Derivative of Inverse Functions. If the given function is expressed as the inverse of $y = f(x)$, namely, $x = \phi(y)$ (see § 48), then for corresponding increments Δx and Δy, provided $\Delta x \neq 0$,

$$\frac{\Delta y}{\Delta x} = \frac{1}{\dfrac{\Delta x}{\Delta y}}.$$

Taking the limits of both sides as Δx approaches zero, we have

$$\text{(IX)} \qquad \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}.$$

That is, *the derivative of y with respect to x is the reciprocal of the derivative of x with respect to y.**

In this case dx/dy, that is $\phi'(y)$, is expressed in terms of y, and its reciprocal dy/dx will be given also in terms of y.

71. Parametric Equations. Suppose x and y are both given in terms of a parameter t, that is,

$$x = g(t), \qquad y = f(t).$$

From § 69 it follows at once that

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}.$$

* Since Formula **V** has been proven valid for n any negative integer (§ 68) as well as positive, now we can show it is valid for n, any rational number. If p and q are any integers, then consider the function,

$$y = u^{1/q}, \qquad \text{where} \qquad u = v^p.$$

Then

$$y^q = u, \qquad \text{and} \qquad y = v^{p/q}.$$

Since u is an integral power of both y and v,

$$\frac{du}{dy} = qy^{q-1} \qquad \text{and} \qquad \frac{du}{dv} = pv^{p-1}.$$

But by **IX**,

$$\frac{dy}{du} = \frac{1}{q}y^{1-q} = \frac{1}{q}(v^{p/q})^{1-q}.$$

Then, since dy/du exists, we have, from **VIII**,

$$\frac{dy}{dv} = \left(\frac{1}{q}v^{p/q-p}\right)\left(pv^{p-1}\right)$$
$$= \frac{p}{q}v^{p/q-1}.$$

Hence for v and y, differentiable functions of x, from **VIII** we have

$$\frac{dy}{dx} = \frac{p}{q}v^{p/q-1}\frac{dv}{dx}, \quad \text{or} \quad \frac{dv^n}{dx} = nv^{n-1}\frac{dv}{dx},$$

where n is any rational number.

Therefore

$$\text{(X)} \qquad \frac{dy}{dx} = \frac{f'(t)}{g'(t)}.$$

This formula will give dy/dx in terms of t.

72. Differentiation of Implicit Functions. Given a function of x and y, say $f(x, y) = 0$. It is possible to find dy/dx without solving the equation $f(x, y) = 0$ for either variable explicitly in terms of the other.

If each term in $f(x, y) = 0$ be differentiated with respect to x, the resulting expression $df/dx = 0$ will contain terms in x, y, and dy/dx. Solving this equation for dy/dx, we have the desired result in terms of x and y.

EXAMPLES

1. Find ds/dt if $s = (1 + r)/(1 - r)$ and $r = \sqrt{2t - t^2}$.

SOLUTION.

$$\frac{ds}{dr} = \frac{(1 - r)(1) - (1 + r)(-1)}{(1 - r)^2} = \frac{2}{(1 - r)^2}.$$

Also

$$\frac{dr}{dt} = \frac{1}{2}(2t - t^2)^{-1/2}(2 - 2t) = \frac{1 - t}{\sqrt{2t - t^2}}.$$

Hence, by **VIII**,

$$\frac{ds}{dt} = \frac{2(1 - t)}{(1 - r)^2\sqrt{2t - t^2}}.$$

2. Find dy/dx when $x = \sqrt{1 + y^2}/(1 - y)$.

SOLUTION. First find dx/dy.

$$\frac{dx}{dy} = \frac{(1 - y)(1/2)(1 + y^2)^{-1/2}(2y) - (1 + y^2)^{1/2}(-1)}{(1 - y)^2}$$

$$= \frac{y(1 - y)/(1 + y^2)^{1/2} + (1 + y^2)^{1/2}}{(1 - y)^2} = \frac{1 + y}{(1 - y)^2(1 + y^2)^{1/2}}.$$

Therefore, by **IX**,

$$\frac{dy}{dx} = \frac{(1 - y)^2(1 + y^2)^{1/2}}{1 + y}.$$

3. Find dy/dx if $x = 3at/(1 + t^3)$ and $y = 3at^2/(1 + t^3)$.

SOLUTION.

$$\frac{dy}{dt} = \frac{(1 + t^3)(6at) - (3at^2)(3t^2)}{(1 + t^3)^2} = \frac{3at(2 - t^3)}{(1 + t^3)^2}.$$

Also

$$\frac{dx}{dt} = \frac{(1 + t^3)(3\,a) - (3\,at)(3\,t^2)}{(1 + t^3)^2} = \frac{3\,a(1 - 2\,t^3)}{(1 + t^3)^2}.$$

Then, by **X**, we have

$$\frac{dy}{dx} = \frac{3\,at(2 - t^3)}{3\,a(1 - 2\,t^3)} = \frac{t(2 - t^3)}{1 - 2\,t^3}.$$

4. Find dy/dx if $x^3 + y^3 - 3\,axy + a^3 = 0$.

SOLUTION. Differentiating implicitly with respect to x, we have,

$$3\,x^2 + 3\,y^2\frac{dy}{dx} - 3\,a\left(x\frac{dy}{dx} + y\right) = 0.$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

PROBLEMS

Find dy/dx if:

1. $x = y(1 + y)^2$. *Ans.* $1/(3\,y^2 + 4\,y + 1)$.
2. $x = (y - 1)^2/y$.

Find dx/dy if:

3. $y = (x^2 - 2\,x)(1 - x^2)^2$. *Ans.* $1/[2(1 - x^2)(5\,x^2 - 3\,x^3 + x - 1)]$.
4. $y = (x^2 - 2)/(1 - x^2)$.

Find dy/dx and dx/dy if:

5. $x = t - t^2$, $y = t + t^2$. *Ans.* $(1 + 2\,t)/(1 - 2\,t)$, $(1 - 2\,t)/(1 + 2\,t)$.
6. $x = t^3 - 3\,t$, $y = 1 - 2\,t^4$.
7. $x = 3 - \theta^2$, $y = 2\,\theta^3$. *Ans.* $-3\,\theta$, $-1/3\,\theta$.
8. $x = t^3$, $y = (1 - t^2)^{3/2}$ at $t = 1$.
9. $x = 2\,t$, $y = 2\sqrt{t^2 - t}$ at $t = 2$. *Ans.* $3/2\sqrt{2}$, $2\sqrt{2}/3$.
10. $x = at$, $y = bt - gt^2/2$, and find t if $dy/dx = 0$.
11. $x^2 + y^2 = a^2$. *Ans.* $-x/y$, $-y/x$.
12. $x^{2/3} + y^{2/3} = a^{2/3}$.
13. $x^2 + xy + y^2 = a^2$. *Ans.* $-(2\,x + y)/(2\,y + x)$, $-(2\,y + x)/(2\,x + y)$.
14. $y^2(2\,a - x) = x^3$ at $x = a$.
15. $x^2y + 4\,a^2y = 8\,a^3$ at $(2\,a, a)$. *Ans.* $-1/2$, -2.
16. $x^2 - 4\,xy + x + y + 3 = 0$.
17. $y^2(x + 2\,y) = x - 2\,y$. *Ans.* $dy/dx = (1 - y^2)/2(1 + xy + 3\,y^2)$.

18. $x - \sqrt{xy} + y = 4.$

19. $x^2y^2 - x/y = 7.$

Ans. $(y - 2\,xy^4)/(x + 2\,x^2y^3)$, $(x + 2\,x^2y^3)/(y - 2\,xy^4)$.

20. $\sqrt{x + y} + \sqrt{x - y} = 1.$

21. $x^2 + (a^2y)^{2/3} = a^2.$ *Ans.* $dy/dx = -3\,xy^{1/3}/a^{4/3}$.

73. Derivative of the Logarithm of a Function. Let the function be $y = \log_a u$ and assign to u an increment Δu, then

$$y + \Delta y = \log_a (u + \Delta u),$$

$$\Delta y = \log_a (u + \Delta u) - \log_a u$$

$$= \log_a \left(1 + \frac{\Delta u}{u}\right).$$

$$\frac{\Delta y}{\Delta u} = \frac{1}{\Delta u} \log_a \left(1 + \frac{\Delta u}{u}\right).$$

To evaluate the limit of the right-hand member as Δu approaches zero, it is written in the following form:

$$\frac{\Delta y}{\Delta u} = \frac{u}{u \cdot \Delta u} \log_a \left(1 + \frac{\Delta u}{u}\right) = \frac{1}{u} \log_a \left(1 + \frac{\Delta u}{u}\right)^{u/\Delta u}.$$

Now

$$\frac{dy}{du} = \frac{1}{u} \lim_{\Delta u \to 0} \left[\log_a \left(1 + \frac{\Delta u}{u}\right)^{u/\Delta u}\right] = \frac{1}{u} \log_a \left[\lim_{\Delta u \to 0} \left(1 + \frac{\Delta u}{u}\right)^{u/\Delta u}\right].$$

But, by § 55(*c*), we have

$$\lim_{\Delta u \to 0} \left(1 + \frac{\Delta u}{u}\right)^{u/\Delta u} = \lim_{x \to 0} (1 + x)^{1/x} = e.$$

Therefore*

$$\frac{d(\log_a u)}{du} = \frac{1}{u} \log_a e.$$

Hence, if u is a differentiable function of x, we have, by **VIII**,

$$\textbf{(XI)} \qquad \frac{d(\log_a u)}{dx} = \frac{1}{u} \log_a e \cdot \frac{du}{dx}.$$

* Certain assumptions are involved in this proof. One is that u is definitely not zero, and must be positive if the function $\log_a u$ is real. Another is that the limit of the logarithm of a function is the logarithm of the limit of the function. This is true under the existing conditions but the proof belongs in a more advanced course. Still another is the actual existence of the $\lim\limits_{\Delta u \to 0} (1 + \Delta u/u)^{u/\Delta u}$ which has already been pointed out in § 55(*c*). It will be understood that the use of formulas **XI** and **XIa** involves these assumptions.

If $y = \log u$, this becomes

$$\textbf{(XI } a\textbf{)} \qquad \frac{d(\log u)}{dx} = \frac{1}{u} \cdot \frac{du}{dx}.$$

NOTE: The proof that Formula **V** is valid for any value of n, irrational as well as rational, depends on Formula **XI *a***, provided the function $y = u^n$ satisfies the restrictions imposed by the use of that formula. (See footnote on p. 98.)

That is, taking the logarithm of both sides, we have

$$\log y = n \log u.$$

Since u is a differentiable function of x, so is $\log u$, also $n \log u$ or $\log y$; hence, differentiating implicitly, we have

$$\frac{1}{y} \cdot \frac{dy}{dx} = n \cdot \frac{1}{u} \cdot \frac{du}{dx},$$

or

$$\frac{dy}{dx} = n \frac{y}{u} \frac{du}{dx} = nu^{n-1} \cdot \frac{du}{dx}.$$

The formulas for differentiating the product of functions, and the quotient of two functions, are readily obtained in a similar manner.

EXAMPLE

Find dy/dx if $y = \log [\sqrt{a^2 + x^2}/(2\,ax - x^2)]$.

SOLUTION. From the fundamental laws of logarithms we can write y as follows:

$$y = (1/2) \log (a^2 + x^2) - \log (2\,ax - x^2).$$

Then, by **XI*a***, we have

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{2\,x}{a^2 + x^2} - \frac{2\,a - 2\,x}{2\,ax - x^2} = \frac{x}{a^2 + x^2} - \frac{2(a - x)}{2\,ax - x^2},$$

or

$$\frac{dy}{dx} = \frac{x^3 + 2\,a^2x - 2\,a^3}{(a^2 + x^2)(2\,ax - x^2)}.$$

74. Derivative of the Exponential Function. Assume the function to be $y = a^u$, where u is any differentiable function of x. Take the natural logarithm of both sides, then

$$\log y = u \log a.$$

Since u is $\log y$ times a constant, du/dy exists, likewise its reciprocal

dy/du, and therefore dy/dx. Then, differentiating with respect to x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{du}{dx}\cdot\log a,$$

and

$$\frac{dy}{dx} = y\log a\frac{du}{dx},$$

or

$$\textbf{(XII)}\qquad \frac{d(a^u)}{dx} = a^u\log a\frac{du}{dx}.$$

If $a = e$, this becomes

$$\textbf{(XII } a\textbf{)}\qquad \frac{d(e^u)}{dx} = e^u\frac{du}{dx}.$$

If $y = e^x$, we have

$$\textbf{(XII } b\textbf{)}\qquad \frac{d(e^x)}{dx} = e^x.$$

Hence the exponential function e^x has the interesting property that its rate of change with respect to the variable is always equal to the value of the function. Likewise the graph of the function e^x has the slope at every point equal to the ordinate of that point.

EXAMPLES

1. $\frac{d}{dx}(3^{x^3/2}) = \log 3\cdot 3^{x^3/2}\cdot 3\,x^2/2 = (3/2)x^2\cdot 3^{\,x^3/2}\cdot\log 3.$

2. $\frac{d}{dx}(e^{1-x^2}) = e^{1-x^2}(-2\,x) = -2\,xe^{1-x^2}.$

PROBLEMS

Find the derivative of each of the following functions with respect to its variable. (Nos. 1–12.)

1. $3\log(x^2+4)$. *Ans.* $6\,x/(x^2+4)$.
2. $\log\sqrt{1-y^2}$.
3. $\log\sqrt{(1-y^2)/(1+y^2)}$. *Ans.* $-2\,y/(1-y^4)$.
4. $\log\sqrt[3]{(t^3-1)(t-t^2)}$.
5. $3\,e^{-t^2}$. *Ans.* $-6\,te^{-t^2}$.
6. 4^{x^2-1}.
7. $e^{3x}-2\,e^{3/x}$. *Ans.* $3\,e^{3x}+6\,e^{3/x}/x^2$.

8. $e^{2x}(9x^2 + 6x - 2)$.

9. $\log_{10}(2x + \sqrt{4x^2 + 1})$. *Ans.* $2\log_{10} e/\sqrt{4x^2 + 1}$.

10. $(\log x^2)^2$.

11. $y = x^x$. (HINT: Take log of both sides.) *Ans.* $x^x(1 + \log x)$.

12. $e^{2y} = (1 + e^{2x})/(1 - e^{2x})$.

Find dy/dx and dx/dy in each of the following cases (Nos. 13–15.)

13. $x = e^{2t}$, $y = te^t$, evaluate at $t = 1$. *Ans.* $1/e$, e.

14. $x = t^2e^t$, $y = t\log t$, evaluate at $t = 1$.

15. $x = 2t^{3/2}$, $y = te^{-t}$. *Ans.* $(1 - t)/(3e^t\sqrt{t})$, $(3e^t\sqrt{t})/(1 - t)$.

Find dy/dx in each of the following cases. (Nos. 16–25.)

16. $y = \log\sqrt{(x + 1)/(x - 1)}$.

17. $y = \log[\sqrt{2x - 3}/(x^2 - x^3)^2]$.
Ans. $dy/dx = (11x^2 - 25x + 12)/x(1 - x)(2x - 3)$.

18. $x = 3^{2y}/\sqrt[3]{1 - 2y^2}$.

19. $y = x\log\sqrt{1 - x}$. *Ans.* $dy/dx = x/2(x - 1) + \log\sqrt{1 - x}$.

20. $y = \log^3(xe^{2x})$.

21. $y = e^{x^2}\log x^2$. *Ans.* $dy/dx = (2e^{x^2}/x)(2x^2\log x + 1)$.

22. $xy = 4\log(xy)$.

23. $e^{xy} - 4xy = 2$. *Ans.* $-y/x$.

24. $x\log y^2 + y\log x^2 = a$.

25. If $x = \sqrt{a^2 - y^2} - a\log[(a + \sqrt{a^2 - y^2})/y]$ find dy/dx.
Ans. $y/\sqrt{a^2 - y^2}$.

ADDITIONAL PROBLEMS

Find the derivative of each of the following functions. (Nos. 1–7.)

1. $y = ax^3 - bx^2 + c/x^3$. *Ans.* $3ax^2 - 2bx - 3c/x^4$.

2. $x = (3y - 4)(2y - y^2)^2$.

3. $s = t(3 - 2t + t^2)$. *Ans.* $ds/dt = 3 - 4t + 3t^2$.

4. $y = x/(x^2 + 4)$.

5. $y = x/(x - \sqrt{x^2 + 4})$.
Ans. $dy/dx = (x + \sqrt{x^2 + 4})/(x - \sqrt{x^2 + 4})\sqrt{x^2 + 4}$.

6. $y = \log(x - \sqrt{x^2 + 4})$.

7. $y = \log\sqrt{(e^x - e^{-x})/(e^x + e^{-x})}$. *Ans.* $dy/dx = 2/(e^{2x} - e^{-2x})$.

Find the derivative of each variable with respect to the other in Nos. 8–11.

8. $xy = (x + y)^2$.

9. $xy = x + e^{-y}$. *Ans.* $dx/dy = (x + e^{-y})/(1 - y)$.

10. $xy + 2x + 3y = 6$, and find what value of x makes $dy/dx = 0$.

11. $x = e^{-t}$, $y = e^{2t+1}$. *Ans.* $dy/dx = -2e^{3t+1}$, $dx/dy = -e^{-3t-1}/2$.

12. Find the slope of $y = xe^x$ at the point where $x = 0$. Also find x where the slope is zero.

13. Find the slope of $x^2 + 3xy + y^2 = 5$ at (1, 1). *Ans.* -1.

14. Draw the curve $y = e^{-x^2}$ using the derivative at each point plotted.

15. The same as Problem 14 for $y = \log x$.

16. The same as Problem 14 for $y = e^{1/x}$.

17. The same as Problem 14 for $y = xe^{-x}$.

18. Show that $y = (a/2)(e^{x/a} + e^{-x/a})$ and $y = a + x^2/2a$ have the same slope at their intersection $(0, a)$. This means the curves are tangent to each other.

Find the derivative of each of the following cases. (Nos. 19–29.)

19. $y = a^{x^2-1}$. *Ans.* $2xa^{x^2-1} \log a$.

20. $y = \log \sqrt[6]{(x-3)/(x+3)}$.

21. $y = 3e^{x-1/x^2}$. *Ans.* $3(1 + 2/x^3)e^{x-1/x^2}$.

22. $y = ax^2a^{x^2}$.

23. $x = a^{ay}$. *Ans.* $a^{ay+1} \log a$.

24. $5^{x^2y} = \log^2 (xy)$.

25. $s = 2^{-2t} - (2t)^2$. *Ans.* $-2^{1-2t} \log 2 - 8t$.

26. $u = e^{1-v} \log \sqrt{v^2 - 9}$.

27. $x^2 + xy - y^2 = 0$. *Ans.* $dy/dx = (2x + y)/(2y + x)$.

28. $x^2 - 2x\sqrt{xy} + 2y\sqrt{xy} - y^2 = 0$.

29. $x = \log t^2$, $y = \log^2 t$ at $t = e$. *Ans.* $dy/dx = dx/dy = 1$.

30. Find the slope of $x^3y^2 + 1 = x + 2y$ at (1, 2).

31. Find the slope and the rate of change of the slope with respect to x for $xy = x - y$. *Ans.* $(1 - y)/(1 + x)$, $2(y - 1)/(1 + x)^2$.

32. Find the slope and the rate of change of the slope with respect to x for $x = 3 - t^2$, $y = 2t^3$ and evaluate each at $t = 2$.

CHAPTER V

SOME APPLICATIONS OF THE DERIVATIVE

75. Tangents. Normals. We have seen (§ 59) that at any point $P_1(x_1, y_1)$ on the graph of $y = f(x)$ the slope of the tangent is the value of the derivative for $x = x_1$. If the derivative has been obtained implicitly, and is expressed in terms of both x and y, then the substitution of x_1 and y_1 for x and y, respectively, will give the slope of the tangent at P_1. That is,

$$\frac{dy}{dx}_{\left\{\substack{x=x_1\\ y=y_1}\right\}} = \frac{dy_1}{dx_1} = m_1 \text{ (a constant).}$$

Then the equation of the tangent at P_1 is (§ 23),

$$y - y_1 = m_1(x - x_1). \tag{1}$$

The ***normal to a curve at a point*** P_1 is the line perpendicular to the tangent at P_1. Hence the slope of the normal is (§ 7)

$$-\frac{1}{\dfrac{dy_1}{dx_1}} = -\frac{dx_1}{dy_1} = -\frac{1}{m_1},$$

and the equation of the normal at P_1 is

$$y - y_1 = -\frac{1}{m_1}(x - x_1). \tag{2}$$

76. Angle of Intersection of Two Curves. By ***the angle of intersection of two curves*** is meant the angle between the tangents to the respective curves at a point of intersection. Let P_1 be a point of intersection of the curves whose equations are $y = f_1(x)$ and $y = f_2(x)$. Then, if the slopes of the tangents at P_1 are m_1 and m_2, the angle of intersection β is (§ 8) such that

$$\tan\beta = \frac{m_1 - m_2}{1 + m_1 m_2}, \tag{3}$$

where m_1 is the slope of the tangent with the greater inclination. Or,

if the inclination of each tangent line is found after the slope is obtained, then (§ 8)

$$\beta = \alpha_1 - \alpha_2. \tag{3 a}$$

EXAMPLES

1. Find the equations of the tangent and the normal to the curve $y = x^2 + 4x + 2$ at the point where the tangent is perpendicular to the line $2x - 4y + 5 = 0$.

SOLUTION. The slope of the given line is $m = 1/2$, hence the slope of the required tangent is -2, and that of the normal is $1/2$. From the equation

$$y = x^2 + 4x + 2$$

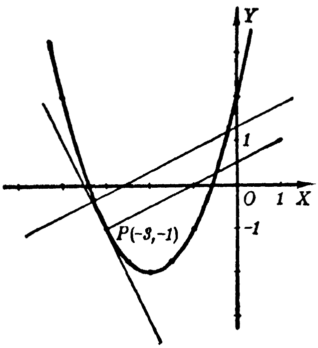


FIG. 83

we have, on differentiating,

$$\frac{dy}{dx} = 2x + 4.$$

The point on the curve which has its slope -2 is located then by

$$2x + 4 = -2.$$

Hence the point of contact desired is $(-3, -1)$, and from **(1)** and **(2)** the tangent and normal are, respectively,

$$2x + y + 7 = 0,$$

and

$$x - 2y + 1 = 0.$$

2. Find the angle between the curves

$$x^2 - 3y = 3 \quad \text{and} \quad 2x^2 + 3y^2 = 30,$$

at their intersection in the second quadrant.

SOLUTION. Solving the equations simultaneously, we find the real intersections are $(3, 2)$ and $(-3, 2)$. Differentiating, we obtain, from the first equation,

$$\frac{dy}{dx} = \frac{2x}{3},$$

and, from the second equation,

$$4x + 6y\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = -\frac{2x}{3y}.$$

Hence, at the point $(-3, 2)$, these slopes are

$$\frac{2x}{3} = -2 = m_1,$$

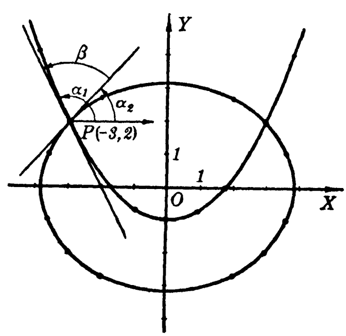


FIG. 84

and

$$-\frac{2x}{3y} = +1 = m_2,$$

respectively. Hence, by (3),

$$\tan\beta = \frac{-2-1}{1-2} = 3, \qquad \beta = 71°\,33.9'.$$

PROBLEMS

Find the equations of the tangent and the normal at the required point on each of the following curves. (Nos. 1–11.) Draw the graph in each case.

1. $xy = 8$, at $(2, 4)$. *Ans.* $2x + y - 8 = 0$, $x - 2y + 6 = 0$.

2. $y = x^2 + 2x + 3$, at the point where the tangent is perpendicular to $x - 2y = 2$.

3. $y = 2 - 3 + 4x^2 - x^3$, where the tangent has the inclination 45°.
Ans. $x - y + 2 = 0$, $27x - 27y + 22 = 0$, $x + y - 6 = 0$, $27x + 27y - 58 = 0$.

4. $y = x^3 - 3x^2 - 6x + 12$, at the point where $x = 2$; at the points where the slope is 3.

5. $y = x^3 + 4x^2$, at $(-1, 3)$.
Ans. $5x + y + 2 = 0$, $x - 5y + 16 = 0$.

6. $x^2 - 2xy + 4y = 0$, where the slope is $-3/2$.

7. $x^2 + 4y^2 = 8$, at the point in the first quadrant where the tangent is parallel to the line through the positive ends of the major and minor axes.
Ans. $x + 2y - 4 = 0$, $2x - y - 3 = 0$.

8. $y = \log x^2$, where the tangent is parallel to $x - 2y + 6 = 0$; perpendicular to $x + y = 1$.

9. $y = x \log x$, where the tangent has slope $3/2$.
Ans. $3x - 2y - 2\sqrt{e} = 0$, $4x + 6y - 7\sqrt{e} = 0$.

10. $y = \log(2x - e)$, at $x = e$.

11. $y = 2e^{-x/3}$, at the crossing of the y axis.
Ans. $2x + 3y = 6$, $3x - 2y + 4 = 0$.

12. For what values of x are tangents on $3y = 4x^2$ and $y = x^3$ perpendicular?

Find the angle of intersection between each of the following pairs of curves. (Nos. 13–21.)

13. $x + y + 2 = 0$, $x^2 + y^2 - 10y = 0$. *Ans.* $\tan^{-1}(1/7)$.

14. $xy = 2$, $x^2 + 4y = 0$.

15. $y^2 = 4x$, $x^2 + y^2 = 5$. *Ans.* $\tan^{-1}(-3)$.

16. $x^2 + 3y = 3$, $x^2 - y^2 + 25 = 0$.

17. $x^2 = 2(y + 1)$, $y = 8/(x^2 + 4)$. *Ans.* $\pi/2$.

18. $2y^2 - 9x = 0$, $3x^2 + 4y = 0$, in the fourth quadrant.

19. $y^2 = 8x$, $4x^2 + y^2 = 32$. *Ans.* $\tan^{-1}(3)$.

20. $x^2y = 4$, $y(x^2 + 4) = 8$.

21. $y = (1/2)e^{-x/2}$ and the y axis. *Ans.* $\tan^{-1}(4)$.

22. Does $y = 3x$ bisect the angle between $y = 2x$ and $y = 4x$? Prove your answer.

23. Show that a tangent to a parabola makes equal angles with its axis and with the line from the focus to the point of contact.

24. Show that the tangent to $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at P_1 is $yy_1^{-1/2} + xx_1^{-1/2} = a^{1/2}$.

25. A tangent to the curve $xy = c$ forms a right triangle with the coordinate axes. Show that the area of this right triangle always has the constant value $2c$ sq. units.

77. Increasing and Decreasing Functions. *A function of a variable is said to be an* ***increasing function*** *if it increases as the variable increases. It is a* ***decreasing function*** *if it decreases as the variable increases.*

Consider the graph of the function, say $y = f(x)$, and trace the curve from left to right so that the variable (or abscissa) is increasing. Then the function (or ordinate) is increasing if the curve is rising; it is decreasing if the curve is falling.

Since the derivative gives the rate of change of the function with respect to the variable, *if the derivative is positive, the function is increasing; if the derivative is negative, the function is decreasing.* Recalling the definition of the derivative, we see immediately that the sign of $\Delta y/\Delta x$ will be positive or negative according as the infinitesimals Δy and Δx have the same or opposite sign. In Fig. 85, any value of x in the interval from $x = a$ to $x = b$ will make the derivative positive and throughout this interval, y increases as x increases. Again, in the interval $x = b$ to $x = c$, the derivative will be negative and throughout this interval, y decreases as x increases. For any value of x for which the function changes from an increasing to a decreasing function, or conversely, the de-

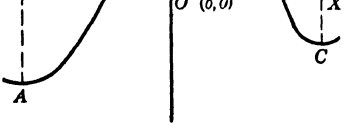


FIG. 85

rivative must change sign; hence it must be zero, if it exists at all, for that value of x. At the corresponding point on the graph of the function, the slope being zero, the tangent is horizontal.

EXAMPLES

1. Find the intervals of the variable x in which the following function $y = f(x) = -3 + 12x - 9x^2 + 2x^3$ increases and decreases, respectively.

SOLUTION. Find the values of x for which the derivative is zero, that is, for which the function may change from increasing to decreasing, or conversely. Now

$$\frac{dy}{dx} = 12 - 18x + 6x^2 = 6(x - 1)(x - 2).$$

When $dy/dx = 0$, $x = 1, 2$. Since dy/dx is continuous, the sign of the derivative can change only at $x = 1$ and $x = 2$. Hence it has one sign in each of the intervals $x < 1$, $1 < x < 2$, and $x > 2$. Try values of x in each interval. Thus for $x = 0$, $dy/dx = 12$ and hence is positive for $x < 1$. Similarly, $x = 1.5$ makes $dy/dx = -3/2$ or $dy/dx < 0$ for $1 < x < 2$. Also $x = 3$ makes $dy/dx = 12$ and so $dy/dx > 0$ for $x > 2$. We may represent this by the following diagram.

$-\infty \longleftarrow$ $x = 1$ $\qquad$ $x = 2$ $\longrightarrow +\infty$

$x < 1$	$1 < x < 2$	$x > 2$
dy/dx positive	dy/dx negative	dy/dx positive
y increasing	y decreasing	y increasing

2. Find the interval of time t in which a body moves in the direction in which its distance s from a fixed point is measured positively, if $s = -20 - 24t + 9t^2 - t^3$.

SOLUTION. As in Example 1, set the derivative equal to zero. Then

$$\frac{ds}{dt} = -24 + 18t - 3t^2 = -3(t - 2)(t - 4) = 0,$$

whence

$$t = 2, 4.$$

Therefore, using $t = 0$, 3, and 5, we get the information shown below.

$-\infty \longleftarrow$ $t = 2$ $\qquad$ $t = 4$ $\longrightarrow +\infty$

$t < 2$	$2 < t < 4$	$t > 4$
ds/dt negative	ds/dt positive	ds/dt negative
s decreasing	s increasing	s decreasing

Hence the body moves in the direction in which s is measured positively in the interval $2 < t < 4$.

PROBLEMS

Find the intervals of the variable in which each of the following functions increases and those in which each decreases. (Nos. 1–7.)

1. $y = x^3 - 3x^2$. *Ans.* Increases $x < 0$, $x > 2$; decreases $0 < x < 2$.

2. $y = x^3(x-2)^2$.

3. $y = x^3 - 3x^2 - 6x + 12$.
Ans. Increases $x < 1 - \sqrt{3}$, $x > 1 + \sqrt{3}$;
decreases $1 - \sqrt{3} < x < 1 + \sqrt{3}$.

4. $s = 2t^3 - 21t^2 + 60t + 5$.

5. $s = t^4/4 - 7t^3/3 - 4t^2 - 2$.
Ans. Increases $-1 < x < 0$, $x > 8$;
decreases $x < -1$, $0 < x < 8$.

6. $y = xe^x$.

7. $s = \log t/t$. *Ans.* Increases $0 < t < e$; decreases $t > e$.

8. If $p = 3v^3 - 7v^2 + 4$, when is dp/dv increasing?

9. If $6s = 2t^3 - 3t^2 + 12t - 4$, where s represents the distance of a particle from a fixed origin, for what interval of t is the particle moving in the direction opposite to which s is measured positively? *Ans.* $1 < t < 2$.

10. The height of a ball is given by $h = 120t - 16t^2$. How long and how high will it rise?

11. The position of a point on a straight line as given by its distance s from some starting point is represented by $s = t^4/4 - 8t^3/3 + 10t^2 - 16t + 7$. When is the motion opposite to the positive direction for s? *Ans.* $t < 4$.

12. Where does the slope of $y = x^4/12 + x^3/6 - x^2 + 3x$ decrease?

13. A particle moves along a line with a velocity given by $v = 1 + 3t^2 - 2t^3$. When is its velocity increasing? When is its acceleration decreasing?
Ans. $0 < t < 1$; $t > 0.5$.

14. A variable rectangle is inscribed in the area bounded by the parabola $x^2 = 8y$ and its latus rectum. One side of the rectangle lies along the latus rectum. As one vertex of the rectangle moves along the curve so that y increases from $y = 0$ to $y = 2$, for what values of y will the area of the rectangle increase?

15. A variable rectangle is inscribed in a circle of radius a units with sides parallel to the reference lines. As a vertex $P(x, y)$ moves along the circle from a position where $x = 0$ to $x = a$, for what values of x will the area of the rectangle decrease? *Ans.* $x > a\sqrt{2}/2$ units.

78. Second and Higher Derivatives. We have found that the derivative of a function of a variable, as $f(x)$, is in general another function of that variable, $f'(x)$. This new function can be differentiated with respect to the variable giving what is known

as the ***second derivative*** of the original function. This we represent by $f''(x)$. Similarly, the derivative of the second derivative, if it exists, is $f'''(x)$, the ***third derivative*** of the original function. Calling the original function y we have the corresponding symbols for the ***higher derivatives.***

$$\text{Second derivative,}\quad f''(x) = \frac{d^2y}{dx^2} = y'' = \frac{dy'}{dx} = D_x^2y.$$

$$\text{Third derivative,}\quad f'''(x) = \frac{d^3y}{dx^3} = y''' = \frac{dy''}{dx} = D_x^3y.$$

From the meaning of the derivative, it is evident that dy'/dx is the rate of change of y' with respect to x. Interpreted on the graph of $y = f(x)$, this means the rate of change of the slope of the tangent with respect to the abscissa of its point of contact. Also (§ 77) y' is an increasing or decreasing function according as y'' is positive or negative. Then as x increases, if y'' is positive the slope of the tangent is increasing; that is, in moving along the curve to the right the tangent will continually turn in a counter-clockwise direction so that the curve will be ***concave upward.*** This is illustrated in Fig. 86 along the arcs A to B, and C to D. If y'' is negative, y' is decreasing; that is, in moving along the curve to the right, the tangent will turn in a clockwise direction so that the curve is ***concave downward.*** This is illustrated along the arcs B to C, and D to E.

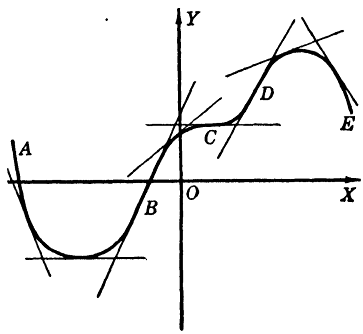


FIG. 86

79. Points of Inflection. If the derivative y' changes from an increasing to a decreasing function as x increases, that is, if the curve changes from concave upward to concave downward, as at B or D in Fig. 86, then the second derivative, y'', changes from a positive to a negative value. Likewise, if y' changes from a decreasing to an increasing function, as at C, then y'' changes from a negative to a positive value. At such points of the curve, B, C, and D, *the second derivative changes sign and becomes zero if it exists; otherwise it may become infinite.* These points on the curve

at which the direction of concavity changes are called ***points of inflection.*** At a point of inflection the tangent crosses the curve, since an arc which is concave upward is above the tangent at any point of the arc, whereas if the arc is concave downward it is below the tangent at any point of the arc.

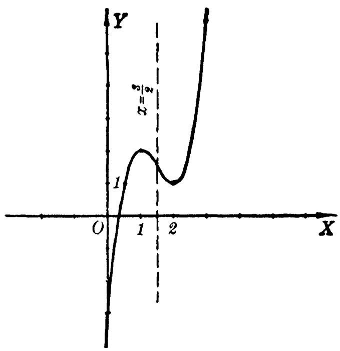


FIG. 87

EXAMPLE

1. If the equation of a given curve is $y = 2x^3 - 9x^2 + 12x - 3$, for what values of x will it be concave downward and for what values will it be concave upward?

SOLUTION. This is the same function which is given in Example 1, § 77. Then $y' = 6(x^2 - 3x + 2)$, $y'' = 6(2x - 3)$. Hence y'' is positive or negative according as $x > 3/2$ or $< 3/2$. Hence the graph is concave downward to the left of $x = 3/2$ and concave upward to the right of $x = 3/2$.

PROBLEMS

Find the second and the third derivatives of each of the following functions. (Nos. 1–6.)

1. $y = x^4/2 - 3x^2$. *Ans.* $6x^2 - 6$, $12x$.

2. $y = x^4/12 + 9x + 4$.

3. $y = e^{-x^2}$. *Ans.* $-2e^{-x^2}(1 - 2x^2)$, $4xe^{-x^2}(3 - 2x^2)$.

4. $y = xe^x$.

5. $y = \log^2 x$. *Ans.* $2(1 - \log x)/x^2$, $2(2\log x - 3)/x^3$.

6. $s = \sqrt{a^2 - t^2}$.

Find d^2y/dx^2 and d^3y/dx^3. (Nos. 7–9.)

7. $x = t - t^2$, $y = t + t^2$. *Ans.* $4/(1 - 2t)^3$, $24/(1 - 2t)^5$.

8. $x = t + 1/t$, $y = t - 1/t$.

9. Find ds/dt, d^2s/dt^2 if $s^2 - s + t^2 = 0$.
Ans. $2t/(1 - 2s)$, $8t^2/(1 - 2s)^3 + 2/(1 - 2s)$.

Find the points of inflection of each of the following curves and observe their intervals of concavity. (Nos. 10–19.)

10. $y = x^3 - 3x^2$.

11. $12y = x^4 - 4x^3 - 18x^2 + 26x + 51$.
Ans. $(-1, 1)$, $(3, -5)$; upward for $x < -1$, $x > 3$, downward for $-1 < x < 3$.

12. $y = x^3 - 4x^2$. Find the direction of the tangent at the inflection.

13. $y = e^{-x^2}$. *Ans.* $(\pm \sqrt{2}/2, e^{-1/2})$; upward $x < -\sqrt{2}/2$, $x > \sqrt{2}/2$, downward $-\sqrt{2}/2 < x < \sqrt{2}/2$.

14. $y = 4x^3 - 6x^2 + 3$. Find the slope of the inflectional tangent.

15. $y = \log(x^2 - 2x + 3)$.
Ans. At $x = 1 \pm \sqrt{2}$; upward $1 - \sqrt{2} < x < 1 + \sqrt{2}$, downward $x < 1 - \sqrt{2}$, $x > 1 + \sqrt{2}$.

16. $y = xe^{-x}$.

17. $y = x/(a^2 + x^2)$.
Ans. At $x = 0$, $\pm a\sqrt{3}$; upward $-a\sqrt{3} < x < 0$, $x > a\sqrt{3}$, downward $x < -a\sqrt{3}$, $0 < x < a\sqrt{3}$.

18. $y = 6x/(x^2 + 1)$.

19. $y = (1/x)\log x$.
Ans. $(e^{3/2}, 3e^{-3/2}/2)$; downward $0 < x < e^{3/2}$, upward $x > e^{3/2}$.

20. Is the curve $y = x^4/2 - 3x^2$ ever concave upward?

21. Discuss the concavity of $y = x^3 - 3x^2 + 4$ at $x = -2, -1, 1, 2$.
Ans. Down, down, neither, upward.

22. Is the curve $y = x^3 - 3x^2 - 20x + 40$ concave upward or downward at $x = -1$? Is y increasing or decreasing at $x = -1$?

23. What are the signs of dy/dx and d^2y/dx^2 for each of the following cases? The curve is (*a*) concave down but rising; (*b*) concave up and rising; (*c*) concave up but falling; (*d*) concave down and falling.

24. Test the curve $y = x^4 - 4x^3 + 6x^2 + 21x - 7$ for points of inflection. How do you explain the result?

80. Maxima and Minima. Let $f(x)$ be a continuous single-valued function of the variable x. By ***single-valued*** we mean that the function has one and only one real value for each value of the variable. If, as x increases, the function y first increases, then decreases, as in Fig. 88 from A to C, there will be one value of the variable, say $x = b$, for which the function is greater than it is for any value of x either a little greater, or a little less than b. Then $f(b)$ is called a ***maximum value*** of the function; that is, MB in Fig. 88 represents a maximum value of $f(x)$.

Similarly, as x increases, if the function decreases, then increases, as from C to E, for some value of the variable, as $x = d$, the function will be less than it is for any value of x either a little greater, or a little less than d. Then $f(d)$ is a ***minimum value*** of the function; that is, ND represents a minimum value of $f(x)$.

If y changes from an increasing to a decreasing function, as when x increases through the value $x = b$, then the derivative

dy/dx must change from a positive to a negative value. But if y changes from a decreasing to an increasing function, as when x increases through $x = d$, the derivative dy/dx must change from a negative to a positive value. At B and at D, where the function assumes its extreme values, the derivative, if continuous, must be zero. We have then the following theorem.

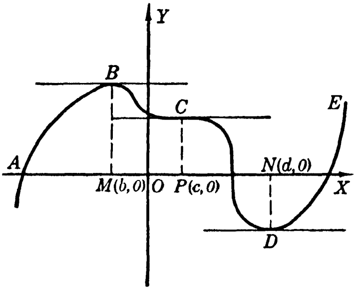


Fig. 88

Given the function $f(x)$, if, as x increases through the value $x = a$, the derivative changes from a positive to a negative value, the function $f(x)$ is a maximum at $x = a$; if the derivative changes from a negative to a positive value, the function $f(x)$ is a minimum at $x = a$.

The derivative may change sign by becoming zero, or by becoming infinite. These two cases are illustrated in Fig. 89. The function is a maximum at A where $y' = 0$, and a minimum at B where y' becomes infinite.

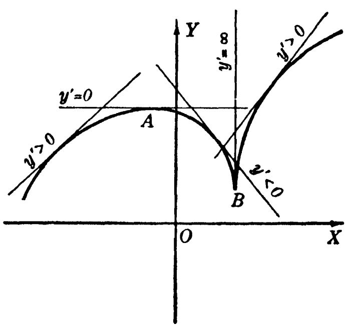


Fig. 89

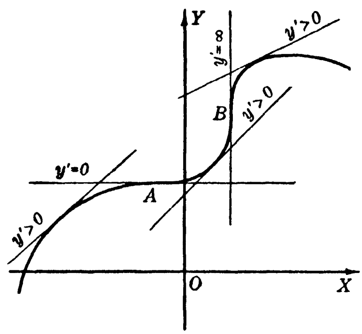


Fig. 90

As x increases, the derivative may become either zero, or infinite, without changing sign. Thus if the graph of the function has a point of inflection with a horizontal tangent, as at A in Fig. 90, both y' and y'' become zero. Here y' decreases until it becomes zero at A, then increases without changing sign. B is a point of inflection with a vertical tangent, that is, y' is infinite, but the sign of y' on either side of B is the same. At such points the function is neither a maximum nor a minimum.

81. Critical Values. Tests for Maxima and Minima. The values of the variable which make the derivative of the function zero, or make the derivative infinite, are called ***critical values*** of the variable. Each value of the variable for which the function is a maximum or a minimum is a critical value; but from § 80 it is evident that the converse is not true. Hence to find the maximum and the minimum values of a function, obtain the critical values of the variable, that is, the values for which $f'(x) = 0$, and those for which $1/f'(x) = 0$, and apply to each value one of the following tests:

FIRST TEST. Let $x = a$ be a critical value. Substitute in the original function $f(x)$ a value of x a little less than a, then again substitute a value a little greater than a. If $f(x)$ in *both* cases is less than $f(a)$, then $f(a)$ is a maximum. If $f(x)$ in *both* cases is greater than $f(a)$, then $f(a)$ is a minimum.

SECOND TEST. Substitute a value of x, first a little less than a, and then a little greater than a in the derived function $f'(x)$, and observe the sign of the derivative in each case. If the sign of the derivative changes from positive to negative, the function is a maximum for $x = a$. If the sign of the derivative changes from negative to positive, the function is a minimum for $x = a$.

In applying either of the tests above for a given critical value, care must be taken that no other critical value lies in the interval between the two values selected for the test.

THIRD TEST. If $f'(x)$ is continuous for $x = a$, when a is a critical value, then substitute a for x in the second derivative, $f''(x)$. If $f''(a)$ is negative, then $f'(x)$ is a decreasing function; but a is a critical value, hence $f'(x)$ decreases through the value zero for $x = a$, and the graph is concave downward. Hence if $f''(a)$ is negative, $f(a)$ is a maximum.

Similarly, if $f''(a)$ is positive when a is a critical value, then $f'(x)$ is increasing through the value zero, the curve is concave upward, and $f(a)$ is a minimum. If $f''(a)$ is zero, the curve usually has a point of inflection at $x = a$, but not always.

The first and second tests may be applied to all critical values of the variable; the third test only to those critical values for which the first derivative is zero. To say that the derivative becomes infinite for some value of the variable, as $x = a$, is merely to say that it increases or decreases without limit and is not defined for that particular value.

EXAMPLES

1. Locate the maximum and minimum points, and points of inflection of $y = x^3 - 3x^2 - 9x - 3$. Trace the graph.

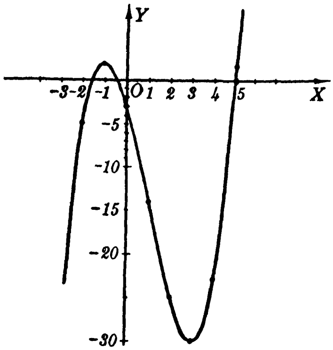


FIG. 91

SOLUTION. Differentiating, we have

$$y' = 3(x^2 - 2x - 3)$$
$$= 3(x-3)(x+1),$$

and
$$y'' = 6(x-1).$$

Set $y' = 0$ to find critical values,

$$x = -1, 3.$$

Set $y'' = 0$ to locate possible points of inflection. This gives

$$x = 1.$$

Here the third test is the simplest.

If $x = -1$, $y' = 0$, y'' is negative, hence $y = f(-1) = 2$ is a maximum. If $x = 3$, $y' = 0$, y'' is positive, hence $y = f(3) = -30$ is a minimum.

Since y'' changes sign at $x = 1$, the point $(1, -14)$ is a point of inflection, with a slope of curve at that point of -12. Note that different scales are used for abscissas and ordinates in Fig. 91 and this must be taken into consideration in estimating the slope at any point.

2. Examine $(x-a)^{1/3}(2x-a)^{2/3}$ for maxima and minima.

SOLUTION. Denoting the function by y, we have $y = (x-a)^{1/3}(2x-a)^{2/3}$, and

$$y' = \frac{(2x-a)^{2/3}}{3(x-a)^{2/3}} + \frac{4(x-a)^{1/3}}{3(2x-a)^{1/3}}$$
$$= \frac{6x - 5a}{3(x-a)^{2/3}(2x-a)^{1/3}}.$$

From $y' = 0$, and $1/y' = 0$, we have the critical values

$$x = \frac{a}{2}, \quad x = \frac{5a}{6}, \quad x = a.$$

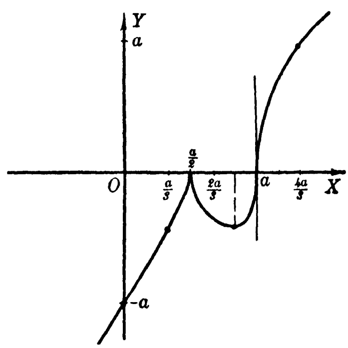


FIG. 92

Use the second test on these values.

Setting $x = a/3$ and $2a/3$ in turn, we have y' positive and negative respectively. Hence $x = a/2$ makes y a maximum.

Test $x = 5a/6$, using the values $2a/3$ and $9a/10$. These show y' to be successively negative and positive, so the function has a minimum value at $x = 5a/6$.

Apply the test to $x = a$, with the values $9a/10$ and $2a$. These show y' positive in both cases and hence at $x = a$, $1/y' = 0$, and the graph of the function has a vertical tangent with neither a maximum nor a minimum. At $x = a$ there is a point of inflection, as shown. The maximum, at the point $(a/2, 0)$, is called a *cusp*.

PROBLEMS

Find the critical values of the variables in each of the following functions. (Nos. 1–16.) Test each and find the maximum and minimum values of the functions.

1. $x^3 - 3x^2$.
   Ans. Max. (0) at $x = 0$. Min. (-4) at $x = 2$.

2. $2x^3 - 3x^2 - 12x + 2$.

3. $x^3(x - 2)^2$.
   Ans. Max. (3456/3125) at $x = 6/5$. Min. (0) at $x = 2$. Inflection at $x = 0$ critical value.

4. $x^2/2 - x^3/3$.

5. $x^4 - 2x^3 + 3$.
   Ans. Min. (21/16) at $x = 3/2$. Inflection at $x = 0$.

6. $6x/(x^2 + 1)$.

7. $3e^{-x^2}$. *Ans.* Max. (3) at $x = 0$. (NOTE: $e^u \neq 0$.)

8. xe^x.

9. $3xe^{-x}$. *Ans.* Max. $(3e^{-1})$ at $x = 1$.

10. $(x + 2)^{2/3}(x - 5)^2$.

11. $3x^5 - 65x^3 + 540x$.
    Ans. Max. at $x = -3, 2$. Min. at $x = -2, 3$.

12. $(2x - a)^{1/2}/(3x + 2a)^{1/3}$.

13. $x/\log x$. *Ans.* Min. (e) at $x = e$. What about $x = 1$?

14. $x^{1/x}$.

15. $ae^{cx} + be^{-cx}$, (a, b positive). *Ans.* $2\sqrt{ab}$ (min.).

16. $3e^{2x} + 5e^{-2x}$.

17. If $y = \Sigma(x - x_i)^2$,* where x_i are constants, $i = 1, 2, \cdots n$, what values of x make y a minimum? *Ans.* $x = x_i$.

18. The equation of the curve described by a jet of water projected from a hose may be represented by $y = kx - (1 + k^2)x^2/100$. What does k represent? What value of k will make the water reach the greatest height on a wall (*a*) 25 feet from the nozzle? (*b*) 45 feet?

19. The total waste per mile in an electric conductor is $w = c^2r + k^2/r$. What resistance r will make the waste a minimum if the current c is kept constant? *Ans.* k/c units.

20. The work done by a voltaic cell of constant E.M.F. and constant internal resistance r in sending a steady current through an exterior circuit of resistance R is $kE^2R/(r + R)^2$ in a given time. What value of R makes the work a maximum? *Ans.* $R = r$.

* Σ here means the sum of n terms formed by giving i consecutive values from 1 to n inclusive.

82. Applications of Maxima and Minima. Many important problems require for their solution that the maximum or minimum value of some quantity be found. Suppose the problem be to find the maximum area of a geometric figure which satisfies given conditions, or to find the minimum amount of material required to build a tank of given capacity, or to find when two moving objects will be nearest together; in all such problems there is a definite method of procedure for finding a solution.

First, the quantity which is to be a maximum or a minimum is always the function to be examined. Hence express the function in terms of the variable or variables which occur in the problem. If the function is obtained in terms of a single variable, the critical values of the variable and the maximum or minimum values of the function can then be found as in the preceding article.

If the function is expressed in terms of two variables, then an additional relation connecting those two variables must be found. Using this relation, one of the variables can be eliminated so that the function will be expressed in terms of a single variable.

Similarly, if the function is given in terms of three variables, two additional relations connecting the three variables must be found. With these relations, two of the variables can be eliminated and the function obtained in terms of a single variable.

After finding the necessary additional relations connecting the variables, the remainder of the solution may be varied as shown in Example 2 below.

EXAMPLES

1. A right circular cone is circumscribed about a sphere of radius a. Find the dimensions of the cone if its volume is a minimum.

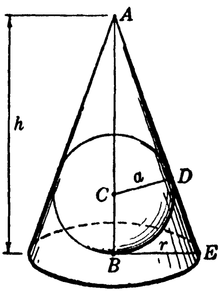


FIG. 93

SOLUTION. Since the volume V is to be a minimum, it is the function in the problem. Hence the function is

$$V = \frac{1}{3}\pi r^2 h. \tag{1}$$

To express the function in terms of a single variable, we must find a relation connecting h and r. Using similar triangles in Fig. 93,

$$\frac{AE}{BE} = \frac{AC}{DC},$$

or

$$\frac{\sqrt{h^2 + r^2}}{r} = \frac{h - a}{a}. \tag{2}$$

Squaring both sides and solving for r^2, we find

$$(3) \qquad r^2 = \frac{a^2h}{h - 2a}.$$

Substituting (3) in (1), we get

$$(4) \qquad V = \frac{\pi a^2h^2}{3(h - 2a)}.$$

Hence

$$(5) \qquad \frac{dV}{dh} = \frac{\pi a^2h}{3}\left[\frac{h - 4a}{(h - 2a)^2}\right].$$

Placing the derivative equal to zero, we find

$$h = 0, \qquad h = 4a.$$

Both the function and its derivative become infinite if $h = 2a$ and hence this value of h cannot be considered. The value $h = 0$ is extraneous since it does not satisfy (2). By applying the second test to the critical value $h = 4a$, we observe that dV/dh is negative for $h < 4a$, and positive for $h > 4a$. Hence $h = 4a$ makes V a minimum. That is, the altitude of the cone is twice the diameter of the sphere and the dimensions of the cone are $h = 4a$, $r = a\sqrt{2}$.

2. What are the most economical proportions for an open cylindrical can of given capacity, if no allowance is made for waste of material?

Solution (*a*). Obviously this means that the volume V of the cylinder is a constant, and that the amount of material M in the can (which forms the lateral surface and one base) is to be a minimum. Then M is the function. Calling the radius r, and the altitude h,

$$(1) \qquad M = \pi r^2 + 2\pi rh,$$

where r and h are connected by the relation,

$$(2) \qquad \pi r^2h = V, \text{ a constant.}$$

Here it is easier to eliminate h, hence from (2) we use

$$(3) \qquad h = \frac{V}{\pi r^2}.$$

Substituting (3) in (1), we find M in terms of r, or

$$(4) \qquad M = \pi r^2 + \frac{2V}{r}.$$

Then

$$(5) \qquad \frac{dM}{dr} = 2\pi r - \frac{2V}{r^2}.$$

Using the value of V from (2), we have

$$(6) \qquad \frac{dM}{dr} = 2\pi r - 2\pi h = 2\pi(r - h).$$

Hence

$$\frac{dM}{dr} = 0 \quad \text{when} \quad r = h.$$

Differentiating (5), we see that $d^2M/dr^2 > 0$ for all possible r. Therefore M is a minimum for an open cylinder with fixed volume if its altitude is equal to its radius.

SOLUTION (*b*). Differentiate implicitly, say with respect to r, both relations (1) and (2); then

$$\frac{dM}{dr} = 2\,\pi r + 2\,\pi\left(h + r\frac{dh}{dr}\right) \tag{7}$$

$$= 2\,\pi\left(r + h + r\frac{dh}{dr}\right),$$

and

$$\pi\left(r^2\frac{dh}{dr} + 2\,rh\right) = 0. \tag{8}$$

Substituting the value of dh/dr from (8) in equation (7), we find

$$\frac{dM}{dr} = 2\,\pi(r + h - 2\,h) = 2\,\pi(r - h). \tag{9}$$

Hence

$$\frac{dM}{dr} = 0 \quad \text{when} \quad r = h.$$

If the independent variable $r < h$, $dM/dr < 0$, and if $r > h$, $dM/dr > 0$, therefore M is a minimum.

3. The amount of fuel consumed per hour by a certain steamer varies as the cube of its speed. When the speed is 15 mi./hr., the fuel consumed is $4\frac{1}{2}$ tons of coal per hour at \$4 per ton. The other expenses total \$100 per hour. Find the most economical speed, and the cost of a voyage of 1980 miles.

SOLUTION. The cost of the voyage C is the function. The cost per hour is $(kv^3 + 100)$ dollars where

$$k = \frac{18}{15^3} = \frac{2}{375}.$$

The time of the trip is s/v where s is the distance. Then

$$C = \left(\frac{2\,v^3}{375} + 100\right)\frac{s}{v} = 2\,s\left(\frac{v^2}{375} + \frac{50}{v}\right),$$

$$\frac{dC}{dv} = 4\,s\left(\frac{v}{375} - \frac{25}{v^2}\right)$$

Equating this to zero, we find the critical value

$$v^3 = (375)(25) = (125)(75).$$

Whence

$$v = 5\sqrt[3]{75} = 21.086 \text{ mi./hr.}$$

$C'' = 4\,s(1/375 + 50/v^3)$ which is positive for all positive values of v, and hence, by the third test, C is a minimum. dC/dv is infinite when $v = 0$, but this makes C infinite. The cost of the trip for $s = 1980$ miles is

$$C = \left(\frac{2\,v^3}{375} + 100\right)\frac{s}{v} = \frac{(150)(1980)}{21.086} = \$14{,}085.$$

PROBLEMS

1. The sum of two positive numbers is 10. Find the numbers if their product is a maximum. *Ans.* 5, 5.

2. The sum of two positive numbers is 12. Find the numbers if the sum of their squares is a minimum.

3. A page of a book must have 18 sq. in. of printed matter and must have 2 in. margins at top and bottom and 1 in. margins on each side. What dimensions will require the least amount of paper? *Ans.* 5 in. by 10 in.

4. What number exceeds its square by the greatest amount?

5. (*a*) A man has 100 rods of fencing and wishes to erect it along three sides of a rectangular field which borders on a straight shore line. What dimensions will give the maximum area? *Ans.* 25 by 50 rods.

(*b*) For any given length of fence, what is the shape of the rectangle?

6. What is the area of the largest isosceles triangle which may be inscribed in the parabolic segment bounded by $y^2 = 8x$ and $x = 8$, if its vertex is at the point (8, 0)?

7. The strength of a rectangular beam varies as the product of its breadth b and the square of its depth h. What is the relation between b and h for the strongest beam which may be cut from a log of radius a units?

Ans. $h = b\sqrt{2}$.

8. A power house on a river bank supplies power to a plant on the other side and 3 mi. down stream. If the river is 2 mi. wide and the power line costs 4/5 as much per mile on land as under water, what line would be cheapest? Would the line under water be changed if the plant were farther down stream?

9. Find the area of the largest rectangle which may be inscribed in a parabolic segment of 30 unit base and 20 unit altitude.

Ans. $400\sqrt{3}$ sq. units.

10. Same as Problem 9 for a semicircle of radius a units.

11. Find the dimensions of the maximum rectangle inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$. *Ans.* $a\sqrt{2}$, $b\sqrt{2}$ units.

12. A rectangular box with square base and cover is to contain 800 cu. ft. If material for the bottom costs 15 cts., for the top 25 cts., and for the sides 10 cts. per square foot, what is the least possible cost of the box?

13. Find the shortest distance from the line $2x + y = 3$ to the point $(-6, 0)$. *Ans.* $3\sqrt{5}$ units.

14. Find the shortest distance from $(-6,0)$ to the hyperbola $x^2 - y^2 + 16 = 0$.

15. A telephone company finds that it makes a net profit of $15 per phone for 1000 phones or less in a given period. If the profit decreases 1 ct. per phone over 1000, what number of phones would yield the greatest profit?

Ans. 1250 phones.

16. What are the dimensions of the largest right circular cylinder which may be inscribed in a sphere of radius a units?

17. (*a*) What is the least material needed to make an open circular cylindrical can of volume 8π cu. in.?

(*b*) For any fixed volume what must be the relation between the radius and altitude of such a can? What if closed at both ends?

Ans. 12π sq. in.; $r = h$, $h = 2r$.

18. The distance of a body from a fixed point is given by the relation

$$s = t^4/12 - 5t^3/6 + 2t^2 + 3t + 1.$$

If the body moves along a straight line, when is it moving most slowly?

19. (*a*) An ellipse of 6 and 8 unit axes is revolved about its major axis. What is the volume of the largest right circular cone which may be inscribed in the solid if its vertex is at an end of the major axis? *Ans.* $128\pi/9$ cu. units.

(*b*) Prove that such an inscribed cone always has its altitude equal to two-thirds of the corresponding axis of the ellipse.

20. (*a*) Find the equation of the line through (4, 3) which cuts off the triangle of least area in the first quadrant.

(*b*) What are the intercepts of such a line if it goes through (a, b)?

21. For a given hypotenuse of $2k$ units, what is the area of the largest right triangle? *Ans.* Area $= k^2$ sq. units.

22. A ship sails south 6 mi./hr. and another east 8 mi./hr. At 4 P.M. the second ship crosses the path of the first at the point where the first was at 2 P.M. When are the ships closest to each other?

23. Given an amount of lumber to make a rectangular box of largest volume. What dimensions should be used if there is no top and if the base dimensions are in the ratio 2 : 1? *Ans.* 6 : 3 : 2.

24. A right triangle with hypotenuse 3 in. is revolved about one leg. What dimensions will the triangle have if the volume generated is a maximum?

25. The intensity of light varies inversely as the square of the distance from its source. If two lights are 300 yds. apart, and one light is 8 times as strong as the other, where should an object be placed between the lights to have the least illumination? *Ans.* 200 yds. from the stronger light.

26. Find the dimensions and volume of the right circular cylinder of largest surface which may be inscribed in a right circular cone of dimensions r and h.

27. What percent of a precious stone, spherical in shape, may be saved if it is cut in the shape of a right circular cone? *Ans.* $29\frac{17}{27}\%$.

28. Same as Problem 27 except that the shape is a regular pyramid with square base.

29. A silo is to be built in the form of a cylinder with a hemispherical roof. The floor and wall are of the same material but the roof costs $2\frac{1}{2}$ times as much per sq. unit as the floor. Find the most economical shape.

Ans. $h = 4r$ for cylinder.

30. Prove that the maximum and minimum line segments from the point (h, k) to the curve whose equation is $y = f(x)$ meet the curve at points where the tangent is perpendicular to the segment.

83. Rates. The essential meaning of the derivative has been shown to be the rate of change of the function with respect to the variable. If the function in question is a linear function, that is, involves the variable to the first degree, as $y = ax + b$, then $dy/dx = a$, or y changes at a constant rate with respect to x. The student is already familiar with the fact that the graph of such a function is a straight line and a, the coefficient of x, is the slope of the line. Thus if r and s are connected by the relation $s = -2r + 5$, then s is decreasing at twice the rate r increases since $ds/dr = -2$, and 5 is the value of the function s when $r = 0$.

If the function $y = f(x)$ is not linear, then the derivative $f'(x)$ is also a variable quantity and will depend for its definite values upon particular values assigned to x. That is, the rate of change of the function with respect to the variable will depend on the variable.

A case of extreme importance is that in which the independent variable is the time t, or else the independent variable is itself a function of the time. If $y = f(t)$, then dy/dt is the rate of change of y per unit time. In many physical problems the time rate of change dx/dt of the independent variable x is given, or can be found, and the time rate of change dy/dt of a related variable y is desired. To solve such a problem, the relation $y = f(x)$ connecting the variables must be obtained; then the desired rate is given at once by the formula

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

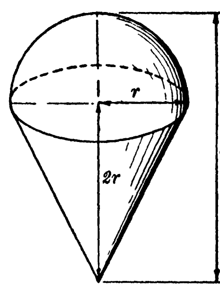


FIG. 94

EXAMPLES

1. A balloon in the form of a right circular cone surmounted by a hemisphere and having its diameter equal to the height of the cone is being inflated. How fast is its volume V changing with respect to its total height h? What is the result when $h = 9$ units?

SOLUTION. Given $h = 3r$, to find dV/dh. Hence we must express V in terms of h. The volume of the cone is $\pi r^2 \cdot 2r/3$, and that of the hemisphere is $2\pi r^3/3$. Adding these, we have

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{h}{3}\right)^3 = \frac{4}{81}\pi h^3.$$

Then

$$\frac{dV}{dh} = \frac{4\,\pi h^2}{27}.$$

Hence V is changing $4\,\pi h^2/27$ times as fast as h. When $h = 9$, $dV/dh = 12\,\pi$, that is, V is then changing $12\,\pi$ times as fast as h.

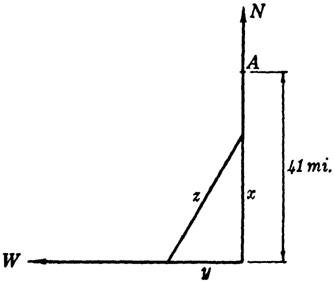


FIG. 95

2. A ship is 41 miles due north of a second ship. The first sails south at the rate of 9 miles per hour, the second sails west 10 miles per hour. (*a*) How rapidly are they approaching each other $1\frac{1}{2}$ hours later? (*b*) How long will they continue to approach each other?

SOLUTION. After t hours of sailing let x be the distance of the first ship from the intersection of the courses, y the distance of the second ship, and z the distance between them. Since x is decreasing 9 mi./hr., and y is increasing 10 mi./hr., we have

$$\frac{dx}{dt} = -9 \text{ mi./hr.}, \qquad \frac{dy}{dt} = 10 \text{ mi./hr.}$$

We require in (*a*) to find dz/dt when $t = 1\frac{1}{2}$; in (*b*) to find what value of t makes z a minimum.

(*a*) To find dz/dt express z in terms of x and y, the variables whose rates are given. Evidently

(1) $$z^2 = x^2 + y^2.$$

Now x, y, and hence z are each functions of t, since

(2) $$x = 41 - 9\,t, \qquad y = 10\,t.$$

Differentiating (1) with respect to t, we find

$$2\,z\frac{dz}{dt} = 2\,x\frac{dx}{dt} + 2\,y\frac{dy}{dt},$$

or

(3) $$\frac{dz}{dt} = \frac{x\dfrac{dx}{dt} + y\dfrac{dy}{dt}}{z} = \frac{-9\,x + 10\,y}{z}.$$

When $t = 1\frac{1}{2}$, from (2) and (1) we have

$$x = 27.5, \qquad y = 15, \qquad z = 31.325.$$

Hence

$$\frac{dz}{dt} = \frac{-247.5 + 150}{31.325} = -3.11 \text{ mi./hr.},$$

that is, in $1\frac{1}{2}$ hours the ships are approaching each other at the rate of 3.11 mi./hr.

An alternative method of solution is to express z directly in terms of t by substituting equations (2) in (1).

(*b*) The ships will continue to approach until z is a minimum, or until

$$\frac{dz}{dt} = \frac{-9x + 10y}{z} = 0,$$

that is, when

$$-9x + 10y = 0.$$

Substituting values of (2), we have,

$$-369 + 81t + 100t = 0, \qquad 181t = 369,$$

or $t = 2.04$ hrs. nearly.

PROBLEMS

1. A spherical balloon is inflated so its volume increases 12 cu. ft./min. How fast is the radius changing when it is 6 ft.? Compare the rates of change of the volume and surface when the radius is 8 ft.

Ans. $1/(12\pi)$ ft./min.; 4 : 1 numerically.

2. If $y = x^3 - 6x^2 + 3x + 5$, at what points are the ordinates and the slopes of the curve changing at the same rate with respect to x?

3. If the amount of wood in a tree is proportional to the cube of its diameter, compare the rates of the growths of two trees of 3 ft. and 6 ft. diameters. *Ans.* 1 : 4.

4. Two ships are at the same point. One leaves at 10 A.M. sailing east 9 mi./hr.; the other at 11 A.M. sailing south 12 mi./hr. How fast are they separating at noon?

5. A point moves along the curve $y = 4 - 2x^2$ so that its abscissa is decreasing 5 units per second. How fast is its ordinate changing as the point passes through (1, 2)? *Ans.* 20 units/sec.

6. The height of a ball thrown upward is given by $h = 120t - 16t^2$. How fast is the ball rising or falling at $t = 3$ sec., 4 sec.? How long does it rise?

7. A man 6 ft. tall walks at 2 ft./sec. toward a light 10 ft. above the ground. How fast is the length of his shadow decreasing? How fast is the end of his shadow moving? *Ans.* 3 ft./sec.; 5 ft./sec.

8. A barge, whose deck is 10 ft. below the level of a wharf, is drawn in by a cable through a ring in the floor of the wharf. A windlass at the level of the deck hauls the cable in 5 ft./sec. How fast is the barge moving toward the wharf when it is 20 ft. away? Is there a maximum velocity for the barge?

9. A conical funnel of height and radius each 6 units contains a liquid which escapes at the rate of 1 cu. unit/min. How fast is the surface falling when it is 4 units from the top of the funnel? *Ans.* $1/(4\pi)$ units/min.

10. In Problem 9, how fast is the inner surface of the funnel being exposed above the liquid?

11. In Problem 9, how fast is the area of the exposed surface of the liquid changing? *Ans.* -1 sq. unit/min.

12. A body is being raised by a rope over a pulley 25 ft. above the body. A man's hand holding the end of the rope is 5 ft. above the body. If the rope is 50 ft. long, at what rate will the body start to rise if the man walks away from under the pulley at 10 ft./min.?

13. In a right triangle the legs are increasing 1 unit/sec. and 2 units/sec. respectively. At what rate is the hypotenuse changing at the time the legs are 3 and 4 units in length? *Ans.* $2\frac{1}{5}$ units/sec.

14. Solve Problem 13 if the first leg increases 1 unit/sec. and the other decreases 3 units/sec.

15. An isosceles triangle has its vertex at $P(x, y)$ a point of the curve $y = e^x$. Its base is along the x axis with one extremity fixed at the origin. If P is moving along the curve so that its ordinate is increasing 5 units/sec., how fast is the area of the triangle changing? *Ans.* $5(x + 1)$ sq. units/sec.

16. In Problem 15 let the vertex P be on the curve $y = x \log x$. Find the rate of change of the area of the triangle and evaluate this for the point (e, e).

17. A man walks across a bridge at the rate of 5 ft./sec. and a boat beneath him passes down stream 12 ft./sec. If the bridge is 30 ft. above the water, how fast are man and boat separating 4 sec. later? What does this rate approach as t increases without limit?

Ans. $338/\sqrt{901}$ ft./sec. $\equiv 11.30$ ft./sec.; 13 ft./sec.

18. A train is moving along an elevated track 20 ft. high at 30 ft./sec. Immediately below it, a truck is going in the same direction 10 ft./sec. How fast are the train and the truck separating one minute after the train is above the truck?

19. The lower end of a ladder 26 ft. long is being pulled away from a vertical wall at 3 ft./sec. How fast is the upper end, resting against the wall, descending when the lower end is 10 ft. from the wall; 24 ft. from the wall? When are both ends moving at the same rate? *Ans.* $1\frac{1}{4}$ ft./sec.; $7\frac{1}{5}$ ft./sec.

20. A trough 8 ft. long has for a cross-section an isosceles trapezoid of altitude 1 ft., upper base 4 ft., lower base 2 ft. If water is poured into the trough at the rate of 2 cu. ft./min., how fast is the depth increasing when it is 6 inches?

21. Water is being poured into a 10 ft. trough at the rate of 25 cu. in./sec. If the ends are isosceles triangles with altitude equal to one half of the base, find the rate of rise of the level of the water when it is 10 inches deep.

Ans. 1/96 in./sec.

22. A balloon is rising vertically 10 ft./sec. from a point on the ground. After 1 min., how fast is it receding from an observer 800 ft. from the point?

23. Water is running out of a horizontal cylindrical tank 9 ft. long and 3 ft. in diameter. When the water is 1 ft. deep, the surface area of the water is decreasing 2 sq. ft./min. At what rate is the depth decreasing?

Ans. $2\sqrt{2}/9$ ft./min.

24. If the circumference of a great circle on a sphere is decreasing 2 in./sec., show that the rate at which the volume of the sphere decreases is numerically equal to the area of the square circumscribing the great circle.

25. A spherical tank of 10 ft. diameter is receiving water at 12 cu. ft./min. At what rate is the depth of the water increasing at 8 ft.? *Ans.* $3/(4\pi)$ ft./min.

26. An arc light is 24 ft. above one side of a street which is 30 ft. wide. A man 6 ft. tall walks along the opposite side at the rate of 5 ft./sec. When he is 40 ft. from the point opposite the light, how fast is the tip of his shadow moving? How fast is his shadow lengthening?

27. The adiabatic law for the expansion of air is $pv^n = c$ where $n = 1.41$, approximately. If the air has a volume of 600 cu. in. at 40 lb. pressure per sq. in., what is the rate of change of the volume with respect to the pressure when $p = 40$ lbs./sq. in.? Approximately how much will the volume be changed due to an increase of 2/3 lb./sq. in. in the pressure?

Ans. $-$ 10.64 cu. in./unit p; 7.09 cu. in. decrease.

28. A point moves along the curve $y = x \log x$ so that y decreases at the rate of 2 units/sec. (*a*) How fast is x changing when the point crosses the line $x = 3y$? (*b*) Find how fast the slope of the graph is changing. (*c*) Show that the abscissa, ordinate, and slope are changing at the same rate when the point crosses the x axis.

84. Rectilinear Motion. One kind of time-rate problem which deserves special mention is rectilinear motion.

Let a particle move along a straight line so that its distance s from a fixed origin on the line is a function of the time t. We have seen (§ 60) that the rate of change of s with respect to t, the time derivative of s, is the ***velocity*** v of the particle. Hence if $s = f(t)$, by differentiating we have

$$v = \frac{ds}{dt} = f'(t).$$

Similarly, the rate of change of v with respect to t, the time derivative of v, is the ***acceleration*** a of the particle. Hence

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t).$$

If the velocity is constant, the motion is called ***uniform.*** If the acceleration is constant the motion is called ***uniformly accelerated motion.*** Thus, a particle near the earth's surface, which is subject to the force of gravity only, moves with an acceleration of $g = 32$ ft./sec.^{2}, approximately.

EXAMPLE

A ball thrown vertically upward has its distance in feet from the starting point given by $s = 104\,t - 16\,t^2$, where t is measured in seconds. Find its velocity, acceleration, and the height the ball will rise. How high is the ball after 3 seconds; after 4 seconds? What distance does the ball pass over during the fourth second?

SOLUTION. The velocity and acceleration are given at once by

$$v = \frac{ds}{dt} = (104 - 32\,t)\text{ft./sec.};$$

$$a = \frac{d^2s}{dt^2} = -32\text{ ft./sec.}^2$$

The ball will rise until s is a maximum, that is, when

$$v = \frac{ds}{dt} = 0.$$

Hence $104 - 32\,t = 0$, $t = 3\frac{1}{4}$ sec. The height it will rise is the value of s for $t = 3\frac{1}{4}$, or

$$s = 169\text{ feet.}$$

For $t = 3$ sec., $s = 168$ ft.; and for $t = 4$ sec., $s = 160$ ft. The distance the ball moves during the fourth second is not Δs for $t = 3$ and $\Delta t = 1$, since the velocity changes sign during the fourth second and hence Δs changes sign. Since we have found the maximum value of s to be 169 ft., we see that during the fourth second the ball rises 1 ft. and falls 9 ft. Therefore the distance traveled during the fourth second is 10 ft.

PROBLEMS

The following laws refer to straight-line motion in each case.

1. A body moves so that $s = t^2 - 8\,t + 7$. When will its velocity be positive? *Ans.* $t > 4$.

2. A body moves with $v = 1 + 3\,t^2 - 2\,t^3$. When is its acceleration decreasing?

3. If $s = 100\,t - 16\,t^2$, when is (*a*) s increasing; (*b*) v decreasing; (*c*) a increasing? *Ans.* $t < 3\frac{1}{8}$; all t; constant.

4. If $s = t^3 - 2\,t^2$, when is v increasing? Is s increasing or decreasing at $t = 1$?

5. What is the direction of motion of a body if its distance s from a fixed point is given by $s = 2\,t^3 - 21\,t^2 + 60\,t + 5$?

Ans. s increasing $2 < t < 5$, decreasing $t < 2$, $t > 5$.

6. The same as Problem 5 if (*a*) $s = 6 + 24\,t - 15\,t^2 - 2\,t^3$, (*b*) $s = t^2 \log t$.

7. The same as Problem 5 if $s = t^3 - 3\,t^2 + 3\,t + 4$.

Ans. Always forward except $v = 0$ at $t = 1$.

8. The motion of a point is determined by $s = -t^2 - 8t + 7$. When is it speeding up?

9. The velocity of a car after t min. is given by $v = t^3 - 21t^2 + 80t$. When is it in reverse? *Ans.* $5 < t < 16$.

10. If $v = t^3 - 5t^2 + 7t - 3$, when is the distance s increasing? How fast does the point move when its acceleration is a maximum or a minimum?

11. If $s = t^4 - 2t^3 - 12t^2 + 36t - 10$, when is s increasing? When is the particle not in motion? When is its velocity decreasing? When is v constant? Has its acceleration an extreme value?
Ans. $-\sqrt{6} < t < 3/2$, $t > \sqrt{6}$; $t = \pm\sqrt{6}$, $3/2$; $-1 < t < 2$; $t = -1, 2$; min. at $t = 1/2$.

12. A particle moves according to each of the following laws. Graph each of the functions s, v, and a against the variable t on the same set of reference lines. Study graphs to obtain data about increasing, decreasing, extreme, and stationary values of s, v, and a.

(*a*) $s = t^3/3 - t^2$. (*c*) $s = 4(4 - e^{-t/2})$.
(*b*) $s = \log\sqrt{2 - 3t}$. (*d*) $s = (\log t)/t$.

ADDITIONAL PROBLEMS

1. Find the tangent and the normal to $xy = 6$ at $(2, 3)$.
Ans. $3x + 2y - 12 = 0$, $2x - 3y + 5 = 0$.

2. Find the tangent and the normal to $y^3 = 2x^2 + 3yx^2$ at $(-1, 2)$.

3. Show that the tangent to $y^2 = 2px$ at $(x_1\ y_1)$ is $y_1y = p(x + x_1)$.

4. (*a*) The line $4x - 3y = 55$ is tangent to the curve whose equation is $3y = x^3 - 3x^2 - 20x + 25$. Find the point of contact.

(*b*) Find the point at which the line $3x + 4y = 54$ is normal to the same curve.

5. Find the line through $(-4, -3)$ which is parallel to the tangent to $y = x^4 - 3x^2 + 23x$ at $x = -2$. *Ans.* $3x - y + 9 = 0$.

6. Find the line through $(-2, -1)$ which is perpendicular to the tangent to $y = x^3 - 2x$ at $x = 1$.

7. Show that the tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point (x_1, y_1) is $x_1x/a^2 + y_1y/b^2 = 1$.

8. Find the area of the triangle formed by the tangent to $y^2 = 9x$ at $(4, 6)$, the normal at the same point, and the x axis.

9. Find the slope of the tangent to $x = at$, $y = bt - (1/2)gt^2$ at any point. Find the value of t which makes the slope zero. Interpret this value.
Ans. $(b - gt)/a$, b/g.

Find the angles between the following pairs of curves. (Nos. 10–12.)

10. $xy = 4$, $x^2 - y^2 = 6$.

11. $4x^2 + y^2 = 52,\ 3y^2 = 16x$. *Ans.* $\tan^{-1}(54/23)$.

12. $x^2 = 4ay,\ y^2 = 8a^3/(x^2 + 4a^2)$.

13. Find the angle between the tangent to $x^2 = 4y + 4$ at $(-2, 0)$ and the line through $(3, -2)$ and $(-2, 0)$. *Ans.* $\tan^{-1}(3/7)$.

14. Find the tangents to $y = 2x^2 - 2$ and $y - x^2 + 1 = 0$ at their intersections and find the angle between them.

15. The curves $y = 2e^{-x/2}$ and $y = \log(x + e)^2$ meet on the y axis. Find the angle of intersection. *Ans.* $\tan^{-1}[(2 + e)/(2 - e)]$.

16. If $P(x, y)$ is on $y = x^3$ and the tangent at P cuts the x axis and the y axis at Q and R, respectively, show that $PR = 3PQ$.

Find the intervals in which each of the following functions increase and those in which they decrease. (Nos. 17–19.)

17. $x^3 + 4x^2$.
Ans. Increases $x < -8/3,\ x > 0$; decreases $-8/3 < x < 0$.

18. $x^4/2 - 3x^2$.

19. $12t^2 - 2t^3 - t^4$.
Ans. Increases $t < -(3 + \sqrt{105})/4,\quad 0 < t < -(3 - \sqrt{105})/4$;
decreases $-(3 + \sqrt{105})/4 < t < 0,\ t > -(3 - \sqrt{105})/4$.

20. If the position of a particle is given by $s = t^3 - 6t^2 + 9t - 12$, when is v increasing, when decreasing?

21. If a point on a line is s units from a starting point, find when its velocity is increasing if:
(*a*) $s = 3 + 4t + 30t^2 + 8t^3 - t^4$;
(*b*) $s = 5t^4 - 14t^3 + 300t^2 + 360t$.
Ans. (*a*) $-1 < t < 5$; (*b*) For all t.

Examine each of the following curves for inflections and types of concavity. (Nos. 22–26.)

22. $y = x^3 - 3x^2 - 6x + 12$.

23. $y = x^3 - x^2 + 6x - 1$.
Ans. $(1/3, 25/27)$; upward $x > 1/3$, downward $x < 1/3$.

24. $y = 2e^{-2x^2}$.

25. $y = 3x^4 - 4x^3 - 6x^2 + 4$.
Ans. $(-1/3,\ 95/27),\ (1,\ -3)$; upward if $x < -1/3,\ x > 1$, downward $-1/3 < x < 1$.

26. $y = \log(x^2 + 1)$.

Locate maximum and minimum values of each of the following functions. (Nos. 27–29.)

27. $4x^3 - 6x^2 + 3$. *Ans.* At $x = 0$, max. (3); at $x = 1$, min. (1).

28. $x^4 - 12x^3 + 36x^2 - 50$.

29. $3x^4 - 4x^3$.
Ans. Min. (-1) at $x = 1$; inflection at critical value $x = 0$.

30. Draw the graphs of $y = x \log x$, and $y = x/\log x$ to the same set of axes. Show that each curve passes through the point of the other at which y is a minimum.

31. Draw a careful graph of $y = x^{1/3}(x + 4)$ between $x = -4$ and $x = 3$. Find high or low points and points of inflection.

Ans. $(-1, -3)$ low point; $(0, 0)$ and $(2, 6\sqrt[3]{2})$ inflections.

32. A potato crop is now 120 bushels and worth \$1 per bushel. If the crop would grow 20 bushels per week and lose 10 cts. per bushel in price, when should they be dug to get the best value?

33. An isosceles triangle with its vertex at $(0, 0)$ and with a horizontal base above the vertex has the ends of its base on $x^2 + 2y = 4$. Find the area of the largest such triangle. *Ans.* $8/3\sqrt{3}$ sq. units.

34. An open box with a square base is to be made with a given inner surface. For a maximum volume, find the relation between the height of the box and the side of the base.

35. A covered box whose base has sides $2 : 1$ is to contain 360 cu. ft. If the bottom costs 4 cts., lid 6 cts., and sides 3 cts. per sq. ft., what are the dimensions for a minimum cost? *Ans.* $6\sqrt[3]{3} \times 3\sqrt[3]{3} \times 20\sqrt[3]{3}/3$ ft.

36. A closed cylindrical vessel is to contain a fixed volume V. (*a*) Find the relation of the radius and the height of the most economical vessel. (*b*) If the curved surface and the ends of this vessel each have a thickness of a units, show that the shape of the vessel should remain unaltered for different values of V.

37. A fixed quantity of metal is to be divided between two molds, one a sphere of radius r, and the other a cube of side s. When will the total surface of these solids have an extreme value?

Ans. Max. if $s = 2r$; min. if $s = 0$ or if $r = 0$.

38. Find the maximum trapezoid which can be inscribed in the ellipse $8x^2 + 9y^2 = 72$ if one base of the trapezoid coincides with the major axis of the ellipse.

39. A body of weight w is dragged along a horizontal plane by a force F whose line of action makes an angle θ with the plane. Find when F is least if $F = mw/(m \sin \theta + \cos \theta)$, where m is the coefficient of friction.

Ans. For $\theta = \tan^{-1} m$.

40. What point on $4y = x^2$ is: (*a*) nearest the point $(0, 4)$? (*b*) nearest the line $x - y = 5$?

41. The perimeter of a sector of a circle is 50 units. What radius will make the area of the sector a maximum? *Ans.* $r = 1/4$ of the perimeter.

42. Water escapes from a conical vessel at the rate of 2 cu. units/sec. and is poured in at the rate of 5 cu. units/sec. The altitude of the vessel is 10 in. and the diameter at the top is 15 in.

(*a*) At what rate is the depth of the water increasing when 4 in. deep?
(*b*) At what rate is the top surface of the water increasing?
(*c*) At what rate is the conical surface being inundated?

43. Find the dimensions of the largest circular cylinder which may be cut from a right circular cone of height h, and radius of base r.

Ans. Alt. of cylinder $= h/3$.

44. At a certain time, the radius of a cylinder is 2 ft. and is increasing at the rate of 1 ft./hr., and the altitude is 4 ft. and is decreasing at the rate of 1 ft./hr. When will the cylinder have a maximum volume?

45. A ship is sailing north at the rate of 10 mi./hr. Another ship 190 miles north of the first ship sails S 60° E at the rate of 15 mi./hr. When are they nearest each other? *Ans.* In 7 hrs.

46. A car running 60 ft./sec. passes directly beneath a balloon at the instant a bomb is released. The height of the bomb after t seconds is $800 - 16\,t^2$; at what rate is the distance between the car and the bomb changing at the end of 5 seconds?

47. A conical funnel loses water so that its depth decreases 2 in./sec. when the water is 6 in. deep. If the funnel is 9 in. deep and 6 in. across the top, how fast is the wet surface of the funnel decreasing at that instant?

Ans. $(8/3)\,\pi\sqrt{10}$ sq. in./sec.

48. Find the rate of change of the total surface of a cylinder if $h = 4\,r$ at the instant r is 4 units and if the volume is decreasing 10 cu. units/min.

49. At a given instant the legs of a right triangle are 4 and 9 units respectively. Assume that the shorter leg is caused to decrease 2 units/sec., and the area to increase 5 sq. units/sec. How is the longer leg changing?

Ans. Increasing 7 units/sec.

50. The velocity of a stream of water issuing from the nozzle of a fountain is given by the formula $v^2 = 2\,gh$, where g is the acceleration of gravity, 32 ft./sec.², and h is the height of the surface of the water above the nozzle. If the surface of the water is falling at the rate of 6 inches per hour, at what rate is v changing when $h = 25$ ft.?

51. Given a semicircle lying above its horizontal diameter. Chords are drawn parallel to the diameter and on each chord as a diameter a circle is drawn. What chord will have the highest point of its circle at a maximum distance from the diameter of the semicircle? *Ans.* $r\sqrt{2}$ long.

52. A rhombus $ABCD$ is made by fastening together with hinges 4 rods of length 13 inches each. If A and C are drawn together at the rate of 5 in./sec., at what rate is the area of the rhombus changing when $AC = 10$ inches?

53. A tank standing on level ground is kept full of water to the depth of a ft.; water issues horizontally from a small hole, at a distance of h ft. below the surface, with the velocity $\sqrt{2\,gh}$ ft./sec. What value of h will make the water strike the ground at the greatest possible distance from the tank?

Ans. $h = a/2$.

54. A cone-shaped container of dimensions a and r units is filled with water. Find the radius of the solid sphere which when placed in the container will displace the greatest amount of water.

Ans. $ar/(\sqrt{a^2 + r^2} - r)$ units.

CHAPTER VI

DIFFERENTIALS. THEOREM OF MEAN VALUE

85. Order of Infinitesimals. We have seen that the increments of the independent variable and of the function, when used in the process of differentiation, are infinitesimals. An important idea in the use of infinitesimals is that of their ***order.***

Given two infinitesimals u and v such that v is a function of u. These infinitesimals are said to be of the ***same order*** provided

$$(1) \qquad \lim_{u \to 0} \frac{v}{u} = k,$$

where k has a finite value not zero.

If the $\lim\limits_{u \to 0} (v/u) = 0$, then v is said to be of a ***higher order*** than u. In general, if

$$(2) \qquad \lim_{u \to 0} \frac{v}{u^n} = k \neq 0,$$

where k is finite, then v is said to be an infinitesimal of the ***nth order*** with respect to u.

If u and v are of the same order, then from relation (1) we may write

$$(3) \qquad v = ku + \delta,$$

where δ is a function of u that approaches zero as u approaches zero.

Dividing both sides of (3) by u and taking limits, we have

$$\lim_{u \to 0} \frac{\delta}{u} = 0.$$

Thus δ is an infinitesimal of higher order than u.

86. Differential of a Function. In any given function, as $y = f(x)$, if $\lim\limits_{\Delta x \to 0} (\Delta y/\Delta x) = f'(x) \neq 0$ for a given value of x, then Δy and Δx are infinitesimals of the same order for that value of x. From the reasoning of the preceding article we can write

$$(1) \qquad \Delta y = \Delta f(x) = f'(x) \cdot \Delta x + \delta,$$

where δ, a function of Δx, is an infinitesimal of higher order than Δx, since

$$\lim_{\Delta x \to 0} \frac{\delta}{\Delta x} = 0.$$

The expression $f'(x) \cdot \Delta x$ in (1) is known as the ***principal part*** of the increment of the function and is called the ***differential of the function.*** Hence:

The differential of a function *is the product of the derivative of the function by the increment of the independent variable.*

The symbol for the differential of a function y is dy, so that

$$(2) \qquad dy = df(x) = f'(x) \cdot \Delta x.$$

If the function equals the independent variable, that is, if $f(x) = x$, then $f'(x) = 1$, and we have

$$(3) \qquad dx = \Delta x.$$

Hence, ***the differential of the independent variable*** *is its increment.* However, the differential of any function, other than a linear function, will differ from the increment of the function.

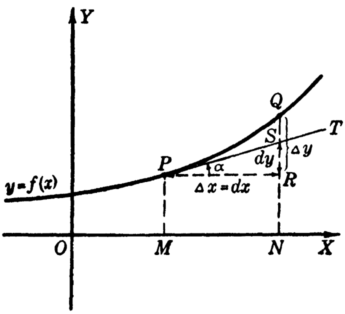


FIG. 96

Let Fig. 96 represent the graph of the function $y = f(x)$. Let $P(x, y)$ be any point of the graph and Q any other point of the graph located by giving x an increment. Draw the tangent PT at P and let α be its inclination. Now the value of the derivative $f'(x)$ at P is the slope of PT, which is the tangent of its inclination. Hence

$$\tan \alpha = f'(x) = \frac{RS}{\Delta x},$$

or

$$RS = f'(x) \cdot \Delta x = dy,$$

since this is the definition of the differential of the function. *However, the increment Δy of the function is RQ.*

Obviously, if P is taken at a point of the graph such that the arc PQ is concave downward, then dy is greater than Δy. It is impor-

tant to keep in mind that for any continuous function the difference between Δy and dy approaches zero as Δx approaches zero; in other words, the difference of Δy and dy is an infinitesimal of higher order than Δx.

Since the differential of a function is its derivative multiplied by the differential of the independent variable, all formulas for differentiation become differential formulas when multiplied by the differential of the independent variable. Thus from

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx},$$

we have

$$d(uv) = u\frac{dv}{dx}dx + v\frac{du}{dx}dx,$$

but by definition

$$\frac{dv}{dx}dx = dv; \quad \frac{du}{dx}dx = du,$$

hence

$$d(uv) = u \cdot dv + v \cdot du,$$

and similarly for each of the formulas for differentiation.

EXAMPLES

1. Find dy if $y = (x - x^2)(2 - 2x - x^2)^{1/2}$.

Solution. Since $dy = f'(x)dx$, we have

$$\begin{aligned} dy &= \left[\frac{-(x - x^2)(1 + x)}{(2 - 2x - x^2)^{1/2}} + (2 - 2x - x^2)^{1/2}(1 - 2x)\right]dx \\ &= (2 - 7x + 3x^2 + 3x^3)(2 - 2x - x^2)^{-1/2}\,dx. \end{aligned}$$

2. If $xy + x^3 = y - 3x/y$ evaluate dy for $x = 1$, and $dx = 0.03$.

Solution. The differential of such an implicit function can be written down at once as follows:

$$x\,dy + y\,dx + 3x^2dx = dy - 3\frac{y\,dx - x\,dy}{y^2}.$$

When $x = 1$, $y = -3$. Substituting these values, and $dx = 0.03$, in the equation above, we get

$$dy - 3(0.03) + 3(0.03) = dy - 3[(-3)(0.03) - dy]/9,$$

or

$$dy = -0.09 \text{ unit}.$$

PROBLEMS

Find the differential of each of the following functions. (Nos. 1–6.)

1. $y = (1 + 2x)/(1 - 2x)$. *Ans.* $4\,dx/(1 - 2x)^2$.

2. $y = 2x\sqrt{4 - x^2}$.

3. $f(s) = (a + s)\sqrt{a - s}$. *Ans.* $\frac{1}{2}[(a - 3s)/\sqrt{a - s}]ds$.

4. $f(s) = \sqrt{(a + bs)/(a - bs)}$.

5. $f(t) = t/\log t$. *Ans.* $df(t) = [(\log t - 1)dt]/\log^2 t$.

6. $f(t) = e^{-t^2}\sqrt{1 - t^2}$.

Determine dy in terms of x, y, and dx from each of the following equations. (Nos. 7–13.)

7. $y(x^2 + 1) = x$. *Ans.* $dy = [(1 - 2xy)/(x^2 + 1)]dx$.

8. $y \log x = x^2$.

9. $x = \log t,\ y = e^{-2t}$. *Ans.* $dy = -2ye^x\,dx$.

10. $x^2/a^2 + y^{2/3}/b^{2/3} = 1$.

11. $(2x + 1)^{2/3}(2y - 1)^{2/3} = 5$.
Ans. $dy = \sqrt[3]{(1 - 2y)/(1 + 2x)}\,dx$.

12. $\log(y/x) - xy = 7$.

13. $e^x - e^y = xy$. *Ans.* $dy = [(e^x - y)/(e^y + x)]dx$.

Evaluate the differential of one variable for the given data in each of the following cases. (Nos. 14–19.)

14. $y = x^3 - x,\ x = 2,\ dx = -0.002$.

15. $2x^2 + 3y^2 = 7,\ x = 1,\ dx = 0.15$. *Ans.* $dy = \pm\sqrt{3}/(10\sqrt{5})$.

16. $\sqrt{x} + \sqrt{y} = 3,\ y = 2,\ dy = -0.2$.

17. $s = t\log t,\ t = e,\ ds = 0.03$. *Ans.* $dt = 0.015$.

18. $x = a^{-y},\ y = a,\ dy = 0.01\,a$.

19. $x = \log t^2,\ y = \log^2 t,\ t = e,\ dt = 0.02\,e$. *Ans.* $dx = dy = 0.04$.

20. If ϵ is an infinitesimal, is each of the following an infinitesimal? (*a*) $3 + 2\epsilon$, (*b*) $\epsilon + 2\epsilon^2$, (*c*) $\sin\epsilon$, (*d*) $\cos\epsilon$, (*e*) $\epsilon/(1 + \epsilon^2)$, (*f*) $\epsilon^2(1 + \epsilon^2)$, (*g*) $\log\epsilon$, (*h*) e^ϵ. Which are of the same order as ϵ? Which of higher order?

21. Given $s = 80t - 16t^2$, calculate the difference between Δs and ds when (*a*) $t = 2,\ dt = 0.1$; (*b*) $t = 4,\ dt = -0.2$. *Ans.* -0.16; -0.64.

22. Given $y = x^3 - x$ and $x = 3$, find the difference between Δy and dy if $\Delta x \equiv dx = 0.02$.

23. If $y = x^2 - 2/x^2$ find $\Delta y - dy$ for $x = 2,\ dx = 0.03$. *Ans.* 0.115.

24. If $s = e^t - \log t$ find $\Delta s - ds$ for $t = 3,\ dt = 0.1$.

87. Approximations. Errors. The differential of a function affords a simple method of approximating the change in a given function due to a small change in the independent variable. Suppose, for instance, it is desired to find the change in the area of a circular metal plate due to expansion caused by a rise in temperature. Simple measurements give the diameter or radius of the plate before and after the change in temperature. Now if A is the area and r the radius of the plate, the problem is to approximate ΔA by finding dA for a given r and Δr. To solve, merely express A as a function of r and find its differential. Then substitute in the expression for dA the values of r and dr, respectively, and obtain the desired approximation.

In many such problems as the one above, it would be absurd to find ΔA by the more cumbersome method of increments, since any values obtained by measurement, as r and dr, are in themselves merely approximations.

All calculations which are based on measurements involve errors and these may be approximated by differentials. Thus if it is assumed that the error in measuring the radius of a circle does not exceed 0.1 inch, then the possible error in the calculation of the area can be approximated by finding dA when $dr = \pm 0.1$ inch.

If x is given an increment, $\Delta x = dx$, in the function $y = f(x)$, the function becomes $y + \Delta y$. However, an approximation of this value by differentials is $y + dy = f(x) + f'(x)dx$.

EXAMPLES

1. Heat applied to a metal plate expands its diameter from 15 inches to 15.14 inches. Approximate the change in area.

SOLUTION. The function is $A = \pi r^2$. To find dA when $r = 7.5$ in. and $dr = 0.07$ in.,

$$\begin{aligned} dA &= 2\,\pi r \cdot dr \\ &= 15\,\pi(0.07) = 1.05\,\pi \text{ sq. in. approximately.} \end{aligned}$$

We note in this example that dA may be interpreted as a rectangular strip of length $2\,\pi r$ and width dr, whereas ΔA is a circular ring of thickness dr bordering a circle of radius r.

2. Find by differentials the reciprocal of 5.03.

SOLUTION. This example involves the reciprocal function,

$$y = \frac{1}{x}.$$

Convenient values for x and dx may be chosen provided dx is relatively small. Since the reciprocal of 5 is 0.2, let $x = 5$ and $dx = 0.03$. By differen-

tials we can find the approximate change in the function due to a change of 0.03 in the variable. Thus

$$dy = -\frac{1}{x^2}dx$$
$$= -\frac{1}{25}\left(0.03\right) = -0.0012.$$

That is, the reciprocal of 5.03 is approximately 0.0012 *less* than the reciprocal of 5. Hence

$$y + dy = 0.2 - 0.0012 = 0.1988.$$

3. Approximate by differentials a root of the equation $x^2 + 3x - 6 = 0$.

SOLUTION. Substituting $x = 0, 1, 2$, we find there is a root such that $1 < x < 2$.

Let $y = x^2 + 3x - 6$. Then

$$dy = (2x + 3)\,dx.$$

For $x = 1$, $y = -2$. But for the root, y should equal 0, whence we let the change in y, namely, $dy = +2$ and we have

$$dx = \frac{dy}{2x+3} = \frac{2}{2\cdot 1 + 3} = 0.4.$$

The new value of x is then

$$x + dx = 1 + 0.4 = 1.4.$$

But $x = 1.4$ makes $y = 0.16$, so taking $dy = -0.16$,

$$dx = \frac{dy}{2x+3} = -\frac{0.16}{2(1.4)+3} = -0.028.$$

The desired approximation is then

$$x + dx = 1.4 - 0.028 = 1.372,$$

which should be compared with the solution by the formula.

88. Relative Error. When errors of measurement are involved in a problem, the ratio of the magnitude of the error to the magnitude of the quantity is usually more significant than the magnitude of the error itself. Evidently, the same actual error made in measuring both a large quantity and a small quantity may be negligible in the former case but not in the latter. If an error Δx is made in the measurement of a given quantity x, then $\Delta x/x$ is called ***the relative error.*** If any function $y = f(x)$ is computed from data which are in error, the ratio dy/y is ***an approximation for the relative error in the function*** and 100 (dy/y) is the ***approximate percentage error.***

The relative error may be approximated directly by logarithmic differentiation since the differential of log y is dy/y.

The applications of the differential may be extended to functions of two or more variables. This, proved in Chapter IX, will be assumed here.

EXAMPLE

If the radius of a right circular cone is measured as 5 in. with a possible error of 0.02 in , and the altitude as 8 in. with a possible error of 0.025 in., what are approximations for the possible relative error, and the possible percentage error in the volume as computed from these measurements?

SOLUTION. We have the values $r = 5$ in., $dr = \pm 0.02$ in., $h = 8$ in., $dh = \pm 0.025$ in. The double sign must be used since the error may be positive or negative. The function is

$$V = \frac{\pi}{3} r^2 h,$$

whence, taking its logarithm and writing the differential, we have

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}.$$

Using the positive values for dh and dr, we may write

$$\frac{dV}{V} = 0.008 + 0.0031 = 0.0111,$$

which is the approximate relative error in V; or

$$100\frac{dV}{V} = 1.11\%,$$

which is the approximate percentage error in V.

If negative values are taken for dr and dh, the results are numerically the same as those above. However, if the values of dr and dh differ in sign, the results are numerically less than those given above. Hence only one set of calculations is necessary to determine the *possible errors*.

PROBLEMS

1. The radius of a circular plate increases by heating from r_0 to $r_0 + \Delta r$. Find an expression in terms of r_0 and Δr for (*a*) the increase in the area; (*b*) an approximation for the increase in the area; (*c*) the error if the result of (*b*) is assumed correct. *Ans.* $\pi(2\, r_0\, \Delta r + \overline{\Delta r}^2)$; $2\, \pi r_0\, \Delta r$; $\pi \overline{\Delta r}^2$ sq. units.

2. The cost of painting a hemispherical dome is 20 cts. per sq. ft. Approximate the error in the estimated cost due to a 3 in. error in measuring the radius as 50 ft. How accurately must the radius be measured if the possible error in the estimated cost must not exceed \$10?

3. Approximate the volume of a right circular cone with a vertical angle of $\pi/2$, if the diameter of its base is 2.9997 units. *Ans.* $1.1247\, \pi$ cu. units.

4. A sphere is to be cut from a cube of edge $2x$. If the diameter of the sphere is to be $2x$ and an error of 1% is made in measuring x, approximate the error in the amount of material to be cut from the cube.

5. Approximate the volume of a right circular cylinder of radius 4.97 in. if its altitude is three times the radius of one end. *Ans.* 368.25π cu. in.

6. How accurately must the diameter of a sphere be measured if it is necessary that the error in the calculated area shall not exceed 0.1%?

7. The edge of a cube is near 6 units. How accurately must it be measured to give an error not to exceed 1 cu. unit in the volume? *Ans.* 1/108 unit.

8. Derive a formula for an approximation of the volume of a thin spherical shell of thickness t. How much is the error in your formula?

9. The hypotenuse and a leg of a right triangle are measured as 5 and 4 inches, respectively. If there is a possible error of 0.01 in. in each measurement, approximate the possible error in the other leg if it is computed from these data. *Ans.* $\pm$ 0.03 in.

10. A central angle is computed from the measurements of the radius and the arc. If 2% errors are possible in each, approximate the possible error in the angle.

11. Approximate the change in the total surface of a right circular cylinder of altitude 10 ft. and radius 4 ft. if its volume is changed 1/2 cu. ft. and the altitude is kept constant. *Ans.* 9/40 sq. ft.

12. How accurately must the altitude of a right circular cone with $r = (4/3)h$ be measured if it is necessary that the percentage error in the calculated volume shall not exceed 3%?

13. If $s = kpv^2$ and p is changed by $+2\%$ and s by -3%, find an approximation for the change in v. Also the relative and percentage changes in v.

Ans. $-0.025v$, -0.025, -2.5%.

14. If the edge of a cube is near 5 in. and there is a possible error of 0.02 in., approximate the resulting relative and percentage errors in the volume and in the surface.

15. A sphere's mass is determined as 1 oz. with a possible error of 0.05 oz. and its diameter as 2 in. with a possible error of 0.02 in. Approximate the possible error in its computed density. *Ans.* $3/5\pi$ units.

Use differentials to approximate the values of each of the following expressions. (Nos. 16–26.)

16. $\sqrt[5]{33}$.

17. $\sqrt[3]{26}$. *Ans.* $2\frac{26}{27}$.

18. $1/\sqrt{50}$.

19. $(123.5)^{4/3}$. *Ans.* 615.

20. $e^{2.01}$, if $e^2 = 7.389$.

21. $\log 10.2$, if $\log 10 = 2.303$. *Ans.* 2.323.

22. $7^{1.98}$, if $\log 7 = 1.946$.

23. $2/[1 + (2.001)^2]$. *Ans.* 0.3999.

24. $x^3 - x$, if $x = 2.002$.

25. $x^4 + 4x^2 + 1$, if $x = 1.997$. *Ans.* 32.856.

26. $x^4 - 2x^3 + 2x^2$, if x is 2 ± 0.015.

Approximate irrational roots of each of the following equations. (Nos. 27–30.)

27. $x^3 - x - 3 = 0$. *Ans.* 1.674.

28. $x^3 - 3x + 5 = 0$.

29. $x^3 + 3x - 10 = 0$. *Ans.* 1.699.

30. $x^3 - x^2 - 5 = 0$.

89. Rolle's Theorem. Let $F(x)$ be a continuous, single-valued function which vanishes for $x = a$ and $x = b$. As x varies from a to b, it is evident that $F(x)$ either increases then decreases, or decreases then increases, so that $F(x)$ will have at least one maximum value or one minimum value between $x = a$ and $x = b$. Now if we add the further condition that $F'(x)$ is continuous in the interval $x = a$ to $x = b$, then $F'(x)$ must become zero at least once in this interval. This gives rise to the following theorem:

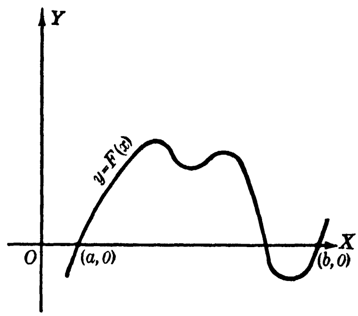


FIG. 97

ROLLE'S THEOREM. *If $F(x)$ is a single-valued function which vanishes for $x = a$ and $x = b$, and if both $F(x)$ and $F'(x)$ are continuous* in the interval $x = a$ to $x = b$, then $F'(x)$ will vanish for at least one value of x in this interval.*

90. Theorem of Mean Value. Let Fig. 98 represent the graph of $y = F(x)$ from $x = a$ to $x = b$. Let R be $[a, F(a)]$ and S be $[b, F(b)]$. Then the slope of the secant RS, which is the average rate of change of the function $F(x)$ from $x = a$ to $x = b$, is

$$\frac{F(b) - F(a)}{b - a}.$$

* It is in fact sufficient to assume that $F'(x)$ exists everywhere, but the theorem as stated is all that we shall need.

The equation of the secant line RS may be written in the form

$$(1) \qquad y = \frac{F(b) - F(a)}{b - a}(x - b) + F(b).$$

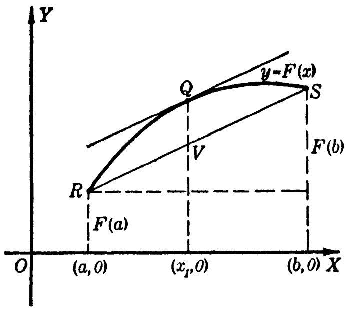


FIG. 98

For any value of x in the given interval, the difference of the two ordinates represented by the y of equation (1), and by $F(x)$ is another function of x, say $\phi(x)$. Then

$$(2) \qquad \phi(x) = F(b) - F(x) + \frac{F(b) - F(a)}{b - a}(x - b)\cdot$$

We see by inspection that $\phi(x)$ vanishes for $x = a$, and for $x = b$. Hence, by Rolle's Theorem, $\phi'(x)$ will vanish for some value of x as x_1 between a and b. That is,

$$(3) \qquad \phi'(x) = -F'(x) + \frac{F(b) - F(a)}{b - a} = 0, \qquad \text{for} \qquad x = x_1,$$

or

$$(4) \qquad F'(x_1) = \frac{F(b) - F(a)}{b - a}, \qquad a < x_1 < b.$$

But $F'(x_1)$ is the slope of the tangent to $y = F(x)$ at the point located by $x = x_1$ and the right-hand member of (4) is the slope of the secant RS. We have then the following theorem:

THEOREM OF MEAN VALUE. *If $F(x)$ is a single-valued function and both $F(x)$ and ~~$F'(x)$~~ are continuous in the interval $x = a$ to $x = b$, then*

$$F(b) - F(a) = F'(x_1)(b - a),$$

where $a < x_1 < b$.

This is also called ***the law of the mean.*** There are two other forms in which this law has important applications. One is obtained by substituting x for b, thus making a variable interval. Then

$$(5) \qquad F(x) = F(a) + F'(x_1)(x - a), \qquad a < x_1 < x.$$

The other is obtained by letting $a = x$, $b = x + \Delta x$, then the interval is Δx, and,

$$(6) \qquad F(x + \Delta x) = F(x) + \Delta x \cdot F'(x + k \cdot \Delta x), \quad 0 < k < 1.$$

91. Extended Theorem of Mean Value. Now let the function be given parametrically by the equations

$$x = g(t), \qquad y = f(t),$$

where $g(t)$, $f(t)$, $g'(t)$ and $f'(t)$ are continuous in the interval $t = a$ to $t = b$. Furthermore, we must assume $g'(t) \neq 0$, so that $dy/dx = f'(t)/g'(t)$ shall exist and be continuous in the interval. The coordinates of R and S are then $[g(a), f(a)]$ and $[g(b), f(b)]$ respectively. Following the method of the preceding section, we find the equation of the secant RS to be

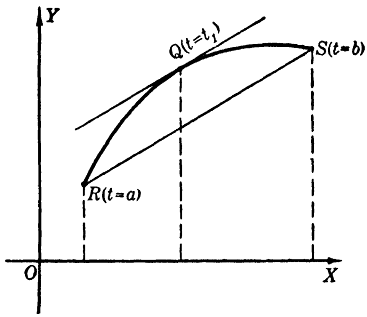


FIG. 99

$$(1) \qquad y = \frac{f(b) - f(a)}{g(b) - g(a)}[x - g(a)] + f(a).$$

For any value of t in the interval, the difference of the corresponding ordinates to the curve and to the secant line is the difference of $f(t)$ and the right-hand member of (1), namely,

$$(2) \qquad f(t) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(t) - g(a)] - f(a).$$

Call this function $\psi(t)$. It vanishes for $t = a$ and for $t = b$; hence, by Rolle's Theorem, $\psi'(t)$ will vanish for some value of t, as $t = t_1$, between a and b, that is

$$(3) \qquad \psi'(t) = f'(t) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g'(t) = 0 \quad \text{for} \quad t = t_1,$$

or

$$(4) \qquad \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(t_1)}{g'(t_1)}, \qquad a < t_1 < b.$$

This is the ***extended theorem of mean value*** or ***extended law of the mean.****** If we make the interval variable by letting $b = t$, it takes the form

$$(5) \qquad \frac{f(t) - f(a)}{g(t) - g(a)} = \frac{f'(t_1)}{g'(t_1)}, \qquad a < t_1 < t.$$

* This theorem is due to the French mathematician Cauchy and (4) is frequently called Cauchy's formula. The geometric proof above was suggested by Professor A. A. Bennett in the *American Mathematical Monthly*, 1924, Vol. 31, p. 41.

The geometric interpretation of the law of the mean or the extended law of the mean is obvious from Figs. 98 and 99, namely, that the slope of the tangent at Q is equal to the slope of the secant RS. Or we can say *if a function and its derivative are continuous in any interval, for some value of the variable within that interval the rate of change of the function will be the same as the average rate of change of the function throughout the interval.*

92. The Indeterminate Form 0/0. If two functions $f(x)$ and $g(x)$ both vanish for $x = a$, their quotient $f(a)/g(a)$ has no meaning. However, the $\lim\limits_{x \to a} [f(x)/g(x)]$ may be perfectly definite. We have already seen this in the case of $(\sin x)/x$ for $x = 0$. Also in the derivative of the function $y = f(x)$, both numerator and denominator of $(\Delta y)/(\Delta x)$ approach zero as a limit, yet the limit of the quotient exists. Such a limit, if it exists, may be found for special cases as in § 55, but a more general method can be obtained from the extended law of the mean. From (5), § 91, we have

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(x_1)}{g'(x_1)}, \qquad a < x_1 < x.$$

But $f(a) = g(a) = 0$; and, since $\lim\limits_{x \to a} x_1 = a$, it follows that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x_1)}{g'(x_1)} = \frac{f'(a)}{g'(a)},$$

if $g'(a) \neq 0$. If both $f'(x)$ and $g'(x)$ are zero for $x = a$, this method may be extended to the form

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \cdots = \lim_{x \to a} \frac{f^n(x)}{g^n(x)},$$

where all derivatives of $f(x)$ and $g(x)$ up to the nth derivative are zero for $x = a$. If either nth derivative is not zero for $x = a$ the limit of the quotient can be evaluated.

EXAMPLES

1. Find $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$.

SOLUTION. This quotient may be found by writing

$$\frac{x^2 - x - 6}{x - 3} = \frac{(x + 2)(x - 3)}{x - 3} = x + 2,$$

only if $x \neq 3$. Using the method of the last section we have

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3} \frac{2x - 1}{1} = 5.$$

2. Evaluate $\lim\limits_{x \to a} \dfrac{a - x}{\log \dfrac{x}{a}}$.

SOLUTION.

$$\lim_{x \to a} \frac{a - x}{\log \frac{x}{a}} = \lim_{x \to a} \frac{a - x}{\log x - \log a} = \lim_{x \to a} \frac{-1}{\frac{1}{x}} = -a.$$

3. Evaluate $\lim\limits_{x \to 1} \dfrac{1 - x + \log x}{x^3 - 3x + 2}$.

SOLUTION. Call the numerator $f(x)$ and the denominator $g(x)$, then

$$\lim_{x \to 1} \frac{f(x)}{g(x)} = \frac{0}{0}, \text{ indeterminate.}$$

$$\lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \frac{-1 + \frac{1}{x}}{3(x^2 - 1)} = \frac{0}{0}, \text{ indeterminate.}$$

$$\lim_{x \to 1} \frac{f''(x)}{g''(x)} = \lim_{x \to 1} \frac{-\frac{1}{x^2}}{6x} = -\frac{1}{6}.$$

93. Other Types of Indeterminate Forms. If the functions $f(x)$ and $g(x)$ are such that $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} g(x) = \infty$, then the fraction $f(x)/g(x)$ assumes the indeterminate form ∞/∞. *The same method is then used as in the case for the indeterminate form* $0/0$. A rigorous demonstration of this fact belongs more properly in an advanced course and will be omitted here.

If $\lim\limits_{x \to a} f(x) = 0$, and $\lim\limits_{x \to a} g(x) = \infty$, then $f(x) \cdot g(x)$ becomes $0 \cdot \infty$ for $x = a$, which is indeterminate. In this case we write the function

$$f(x) \cdot g(x) = \frac{f(x)}{1/g(x)}, \quad \text{or} \quad \frac{g(x)}{1/f(x)},$$

so that the transformed expression takes one of the forms $0/0$ or ∞/∞, and its limit, if it exists, can be evaluated by the method explained in § 92.

An exponential function may assume one of the indeterminate forms 0^0, ∞^0, or 1^∞ for some value of the variable. In this case the

logarithm of the function may assume the form $0 \cdot \infty$ and its limit, if it exists, can be evaluated. (See footnote, § 73.)

The difference of two functions for some value of the variable may assume the indeterminate form $\infty - \infty$. In this case a transformation may be found which will reduce the function to one of the other types of indeterminate forms which have been mentioned.

EXAMPLES

1. Find the $\lim\limits_{x\to\infty} \dfrac{x \log x}{x + \log x}$.

SOLUTION. This function has the indeterminate form ∞/∞.

$$\lim_{x\to\infty} \frac{x \log x}{x + \log x} = \lim_{x\to\infty} \frac{1 + \log x}{1 + 1/x} = \infty,$$

since the denominator has the limit 1, while the numerator increases without limit.

2. Evaluate $\lim\limits_{x\to 1} x^{1/(1-x)}$.

SOLUTION. Call this function $f(x)$, then $\log f(x) = [1/(1 - x)] \log x$. Here $f(x)$ has the indeterminate form 1^∞, for $x = 1$; and $\log f(x)$ has the form $\infty \cdot 0$. Writing $\log f(x)$ as $(\log x)/(1 - x)$, we have

$$\lim_{x\to 1} \frac{\log x}{1 - x} = \lim_{x\to 1} \frac{\frac{1}{x}}{-1} = -1.$$

Hence

$$\lim_{x\to 1} \log f(x) = -1, \qquad \text{or} \qquad \lim_{x\to 1} x^{1/(1-x)} = e^{-1} = 1/e.$$

3. Find $\lim\limits_{x\to 0} \left(\dfrac{1}{e^x - 1} - \dfrac{1}{x}\right)$.

SOLUTION. This function has the indeterminate form $\infty - \infty$ for $x = 0$. Write it in the form $(x - e^x + 1)/x(e^x - 1)$. Then

$$\lim_{x\to 0} \frac{x - e^x + 1}{x(e^x - 1)} = \lim_{x\to 0} \frac{1 - e^x}{xe^x + e^x - 1} = \frac{0}{0}.$$

Differentiating again, we get

$$\lim_{x\to 0} \frac{1 - e^x}{xe^x + e^x - 1} = \lim_{x\to 0} \frac{-e^x}{xe^x + 2e^x} = \lim_{x\to 0} \frac{-1}{x + 2} = -\frac{1}{2}.$$

PROBLEMS

Evaluate the limit, if it exists, of each of the following cases. (Nos. 1–19.)

1. $\lim\limits_{x\to 2} [(x^2 - 2x)/(x^2 - 4)]$. *Ans.* 1/2.

2. $\lim\limits_{x\to\infty} [(x - 2)/(x^2 + 4)]$.

3. $\lim\limits_{x\to\infty} [(5x - 4x^2)/(2x^2 - x + 4)]$. *Ans.* -2.

4. $\lim\limits_{x\to 0}[x/(1-e^x)]$.

5. $\lim\limits_{x\to 0}[(3x-4x^2)/(2x^2+e^x-1)]$. *Ans.* 3.

6. $\lim\limits_{x\to 0}[x\log x]$.

7. $\lim\limits_{x\to 1}[(x^m-1)/(x^n-1)]$. *Ans.* m/n.

8. $\lim\limits_{x\to\infty}[(\log x)/x^2]$.

9. $\lim\limits_{x\to 1}[(\log x^2)/(x-1)]$. *Ans.* 2.

10. $\lim\limits_{x\to\infty}[\{\log(x-a)\}/(x-a)]$.

11. $\lim\limits_{x\to 1}[(e^{x-1}-a^{x-1})/(x^2-1)]$. *Ans.* $(1-\log a)/2$.

12. $\lim\limits_{x\to 1}\left[\dfrac{1}{\log x}-\dfrac{x}{x-1}\right]$.

13. $\lim\limits_{x\to 0}[(1+kx)^{1/x}]$. *Ans.* e^k.

14. $\lim\limits_{x\to\infty}\left[\left(1+\dfrac{1}{x}\right)^{kx}\right]$.

15. $\lim\limits_{x\to 2}\dfrac{\sqrt{3x}-\sqrt{8-x}}{3x-2\sqrt{15-3x}}$. *Ans.* $1/2\sqrt{6}$.

16. $\lim\limits_{x\to 0}[x^n\log x^{2n}]$.

17. $\lim\limits_{x\to 0}[(1+ax)^{(b-cx)/x}]$. *Ans.* $e^{a/b}$.

18. $\lim\limits_{x\to e}[(\log x)^{k/(1-\log x)}]$.

19. $\lim\limits_{x\to 0}[(e^{ax}+bx)^{c/x}]$. *Ans.* $e^{c(a+b)}$.

20. Given the curve whose equation is $y=(1+x)^{1/x}$. Prove that the limiting value of the slope of the tangent as the point of contact approaches the y axis is $-e/2$.

ADDITIONAL PROBLEMS

1. If the edge of a cube is 3 in. and is expanded to 3.002 in., approximate the volume. *Ans.* 27.054 cu. in.

2. Approximate the volume of a right circular cone of $h=2r$ if r is measured as 4.98 in.

3. A sphere of radius 2 ft. has its radius reduced 3 in. Approximate the resulting change in its surface. What is the exact change?

Ans. -4π sq. ft.; $-3\frac{3}{4}\pi$ sq. ft.

4. A spherical casting of $r=1$ ft. is smoothed down so that the radius is decreased 0.1 in. Approximate the volume removed. What is the error in the approximation?

5. Show that $(x + dx)^2$ is approximately equal to $x^2 + 2x\,dx$.

Derive formulas similar to the one in Problem 5 to approximate each of the following expressions. (Nos. 6–10.)

6. $1/(x + dx)$.

7. $\sqrt{x + dx}$. *Ans.* $\sqrt{x} + dx/(2\sqrt{x})$.

8. $\sqrt[3]{(x + dx)^2}$.

9. $\log(x + dx)$. *Ans.* $\log x + dx/x$.

10. $a^{(x+dx)}$.

Approximate the irrational roots of each of the following equations. (Nos. 11–16.)

11. $x^3 - x^2 + 5 = 0$. *Ans.* -1.43.

12. $x^3 - 4x^2 - 2x + 8 = 0$.

13. $x^3 - 3x + 1 = 0$. *Ans.* -1.88, 0.42, 1.59.

14. $x^3 - 8x + 2 = 0$.

15. $x^3 + x - 4 = 0$. *Ans.* 1.38.

16. $x^4 - 11x^2 + 24 = 0$.

17. Two iron spheres are each approximately 10 inches in diameter. When immersed in a pail of water it is found that one sphere displaces 20 cu. in. more of water than the other. Approximate the difference in their radii *Ans.* $1/(5\pi)$ in.

18. What is the percentage error in u if there is an error of 1 in the fourth decimal of $\log_{10} u$?

19. What error in the common logarithm of a number will be produced by an error of 1% in the number? *Ans.* 0.0043.

20. The motion of an object is given by $s = t^3 + 4t^2 - 3t + 5$. Approximate the distance moved from $t = 2$ to $t = 2.02$ sec.

21. The same as Problem 20 except from $t = 1$ to $t = 2.998$ sec. *Ans.* 51.904 units.

22. A point moves along the curve $x = 3\sqrt[3]{t}$, $y = 1 \pm 2\sqrt{t}$. Find the rectangular equation of its path and approximate the change in y when x changes from 9 to 8.97.

23. Rectangles with sides parallel to the two axes are inscribed in the area bounded by $y^2 = 6x$ and its latus rectum. The value of y for one of these rectangles is measured as 1.5 units with a possible error of 0.04 unit, approximate the possible error resulting in its area. *Ans.* 0.03 sq. units.

24. If $s = kp^{1/2}v^2$, approximate the relative change and the percentage change in s for 3% change in v and $1\frac{1}{2}$% change in p.

CHAPTER VII

TRIGONOMETRIC FUNCTIONS — CURVATURE

94. Derivative of sin u. Let $y = \sin u$. Give the independent variable u an increment; then

$$y + \Delta y = \sin (u + \Delta u),$$

and

$$\Delta y = \sin (u + \Delta u) - \sin u.$$

Using the formula

$$\sin A - \sin B = 2 \cos\left[\frac{A+B}{2}\right]\sin\left[\frac{A-B}{2}\right],$$

we get

$$\Delta y = 2 \cos\left(u + \frac{\Delta u}{2}\right) \sin \frac{\Delta u}{2}.$$

Then

$$\frac{\Delta y}{\Delta u} = \frac{2 \cos\left(u + \frac{\Delta u}{2}\right) \sin \frac{\Delta u}{2}}{\Delta u}$$

$$= \cos\left(u + \frac{\Delta u}{2}\right) \cdot \frac{\sin \frac{\Delta u}{2}}{\frac{\Delta u}{2}}.$$

Let $\Delta u \rightarrow 0$; then

$$\lim_{\Delta u \to 0} \left(u + \frac{\Delta u}{2}\right) = u, \qquad \text{and, by § 55,} \qquad \lim_{\Delta u \to 0} \frac{\sin \frac{\Delta u}{2}}{\frac{\Delta u}{2}} = 1.$$

Hence

$$\frac{dy}{du} = \cos u.$$

If u is a differentiable function of x, since

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

then

$$\textbf{(XIII)} \qquad \frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}.$$

95. Derivative of cos *u*. The derivative of $\cos u$ may be obtained by using either the relation $\cos u = \sin(\pi/2 - u)$ or $\cos u = \sin(\pi/2 + u)$. Using the former, we have

$$\begin{aligned}\frac{d}{dx}(\cos u) &= \frac{d}{dx}[\sin(\pi/2 - u)] \\ &= \cos(\pi/2 - u) \cdot \frac{d}{dx}(\pi/2 - u) \\ &= \cos(\pi/2 - u)\left(-\frac{du}{dx}\right).\end{aligned}$$

Therefore

$$\textbf{(XIV)} \qquad \frac{d}{dx}(\cos u) = -\sin u \cdot \frac{du}{dx}.$$

96. Derivative of tan *u*, ctn *u*. To differentiate the tangent function, write $\tan u = \sin u/\cos u$. Then using the formula for differentiating a quotient, we have

$$\begin{aligned}\frac{d}{dx}(\tan u) &= \frac{\cos u \cdot \dfrac{d}{dx}(\sin u) - \sin u \cdot \dfrac{d}{dx}(\cos u)}{\cos^2 u} \\ &= \frac{\cos^2 u + \sin^2 u}{\cos^2 u} \cdot \frac{du}{dx} \\ &= \frac{1}{\cos^2 u} \cdot \frac{du}{dx}.\end{aligned}$$

Hence

$$\textbf{(XV)} \qquad \frac{d}{dx}(\tan u) = \sec^2 u \cdot \frac{du}{dx}.$$

In a similar manner we derive the formula

$$\textbf{(XVI)} \qquad \frac{d}{dx}(\text{ctn } u) = -\csc^2 u \cdot \frac{du}{dx}.$$

97. Derivative of sec *u*, csc *u*. Write $\sec u = 1/\cos u$. Then

$$\frac{d}{dx}(\sec u) = -\frac{\dfrac{d}{dx}(\cos u)}{\cos^2 u} = \frac{\sin u}{\cos^2 u} \cdot \frac{du}{dx}.$$

That is,

$$\textbf{(XVII)} \qquad \frac{d}{dx}(\sec u) = \sec u \cdot \tan u \cdot \frac{du}{dx}.$$

In like manner we find

$$\text{(XVIII)} \qquad \frac{d}{dx}(\csc u) = -\csc u \cdot \operatorname{ctn} u \cdot \frac{du}{dx}.$$

EXAMPLES

1. Find dy/dx if $y = (1/3)\sin 2x - \tan^2 3x$.

SOLUTION.

$$\frac{dy}{dx} = \frac{1}{3}(\cos 2x)(2) - 2(\tan 3x)(\sec^2 3x)(3)$$

$$= \frac{2}{3}\cos 2x - 6\tan 3x \sec^2 3x.$$

2. If $r = [\sec^4(\theta/2) - \cos^2(\theta/2)]\operatorname{ctn}(\theta/2)$, find $dr/d\theta$.

SOLUTION. This function is of the form $u \cdot v$. Hence,

$$\frac{dr}{d\theta} = \left(\sec^4\frac{\theta}{2} - \cos^2\frac{\theta}{2}\right)\left[\left(-\csc^2\frac{\theta}{2}\right)\left(\frac{1}{2}\right)\right]$$

$$+ \operatorname{ctn}\frac{\theta}{2}\left[\left(4\sec^3\frac{\theta}{2}\sec\frac{\theta}{2}\tan\frac{\theta}{2}\right)\left(\frac{1}{2}\right) - 2\cos\frac{\theta}{2}\left(-\sin\frac{\theta}{2}\right)\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\operatorname{ctn}^2\frac{\theta}{2} - \frac{1}{2}\sec^4\frac{\theta}{2}\csc^2\frac{\theta}{2} + 2\sec^4\frac{\theta}{2} + \cos^2\frac{\theta}{2}.$$

PROBLEMS

Differentiate each of the following functions with respect to its variable. (Nos. 1–16.)

1. $y = \operatorname{ctn}(x/3)$. *Ans.* $-(1/3)\csc^2(x/3)$.

2. $y = \cos^3 x^2$.

3. $y = \sin^2 5t$. *Ans.* $5\sin 10t$.

4. $s = \tan^2 2t$.

5. $x = \sin(2 - 3y)\cos(2y - 1)$.
Ans. $-2\sin(2-3y)\sin(2y-1) - 3\cos(2-3y)\cos(2y-1)$.

6. $y = \sin^3 2x \cos 3x$.

7. $y = \sin^2 x \cdot e^{\cos 2x}$. *Ans.* $(1/2)\sin 4x \cdot e^{\cos 2x}$.

8. $y = \log \sin^2(3x/2)$.

9. $s = \sin(\log t)$. *Ans.* $(1/t)\cos(\log t)$.

10. $y = \log(\sec 3x + \tan 3x)$.

11. $y = \log\{[1 - \sin(x/2)]/[1 + \sin(x/2)]\}$. *Ans.* $-\sec(x/2)$.

12. $y = (\tan x)/x^2$.

13. $y = 4\sin(x/2) - 2x\cos(x/2)$. *Ans.* $x\sin(x/2)$.

14. $y = \sqrt{\dfrac{\sec x + \tan x}{\sec x - \tan x}}$.

HINT: $(\sec x + \tan x)$ and $(\sec x - \tan x)$ are reciprocals. Why?

15. $y = 2 \sin^2 (x/2) \cos (x/2)$.

Ans. $\sin (x/2) [2 \cos^2 (x/2) - \sin^2 (x/2)]$.

16. $y = \csc^2 (1 - 2x) \operatorname{ctn} (1 - 2x)$.

Find the derivative of one variable with respect to the other in each of the following cases. (Nos. 17–26.)

17. $xy = \sin 3x$. *Ans.* $dy/dx = (1/x)(3 \cos 3x - y)$.

18. $xy = e^{\cos 2x}$.

19. $\cos (x + y) + \cos (x - y) = 0$. *Ans.* $dy/dx = -\tan x \operatorname{ctn} y$.

20. $xy + \operatorname{ctn} xy = 0$.

21. $\sin x \cos y + \cos 2x \sin 2y = 1$.

Ans. $dy/dx = (2 \sin 2x \sin 2y - \cos x \cos y)/(2 \cos 2x \cos 2y - \sin x \sin y)$.

22. $r^2 \sin 3\theta = a \cos \theta$. Find $dr/d\theta$.

23. $y = \log \sqrt{\dfrac{1 - \cos \theta}{1 + \cos \theta}}$. *Ans.* $\csc \theta$.

24. $r \sec \theta = \sin^2 \theta$.

25. $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$.

Ans. $dy/dx = (\sin \theta)/(1 + \cos \theta) = \tan (\theta/2)$.

26. $x = a \cos \theta + a\theta \sin \theta$, $y = a \sin \theta - a\theta \cos \theta$.

Find dy/dx and d^2y/dx^2 for each of the following cases. (Nos. 27–31.)

27. $x = 2 \sin t$, $y = 2 \cos t$. *Ans.* $-\tan t$, $-(1/2) \sec^3 t$.

28. $x = 2 \sin t + 3 \cos t$, $y = \sin t$.

29. $x = a \tan \theta$, $y = b \sin \theta$. *Ans.* $(b/a) \cos^3 \theta$, $-(3b/a^2) \cos^4 \theta \sin \theta$.

30. $x = a \sin^3 \theta$, $y = a \cos^3 \theta$.

31. $x = e^{2t}$, $y = 2 + \cos 4t$.

Ans. $-(2 \sin 4t)/e^{2t}$, $4(\sin 4t - 2 \cos 4t)/e^{2t}$.

32. Derive the formula for differentiating $\cos u$ with respect to x by increments.

33. The same as Problem 32 for $\tan u$.

98. Derivatives of the Inverse Trigonometric Functions. Let

$$y = \sin^{-1} u$$

where u is a differentiable function of x, then

$$u = \sin y.$$

Differentiating u with respect to y, we have

$$\frac{du}{dy} = \cos y = \pm \sqrt{1 - \sin^2 y} = \pm \sqrt{1 - u^2}.$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \pm \frac{1}{\sqrt{1 - u^2}} \cdot \frac{du}{dx}.$$

Observe that the double sign enters in the result. This is because y is a many-valued function of u in $y = \sin^{-1} u$, whereas u is a single-valued function of y in $u = \sin y$. Since $du/dy = \cos y$, and the smallest values of y for which $\cos y$ is positive lie between $-\pi/2$ and $\pi/2$, we shall ordinarily restrict the values of y to satisfy $-\pi/2 \leqq y \leqq \pi/2$, so that the derivative will not be negative. Then

$$\text{(XIX)} \qquad \frac{d}{dx}(\sin^{-1} u) = \frac{\frac{du}{dx}}{\sqrt{1 - u^2}}.$$

Differentiating $y = \cos^{-1} u$ in a similar manner, we find

$$\frac{du}{dy} = -\sin y.$$

Here again y is a many-valued function of u and if we restrict y to an interval for which $\sin y$ is positive or zero, namely, $0 \leqq y \leqq \pi$, the derivative will then be negative or zero, and

$$\frac{dy}{du} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - u^2}},$$

or

$$\text{(XX)} \qquad \frac{d}{dx}(\cos^{-1} u) = -\frac{\frac{du}{dx}}{\sqrt{1 - u^2}}.$$

If $y = \tan^{-1} u$, $u = \tan y$, and $du/dy = \sec^2 y$, then

$$\frac{dy}{du} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + u^2}.$$

Hence

$$\text{(XXI)} \qquad \frac{d}{dx}(\tan^{-1} u) = \frac{\frac{du}{dx}}{1 + u^2}.$$

For $y = \text{ctn}^{-1} u$, $du/dy = -\csc^2 y$. Hence

$$\frac{dy}{du} = -\frac{1}{\csc^2 y} = -\frac{1}{1 + \text{ctn}^2 y} = -\frac{1}{1 + u^2}.$$

Therefore

$$\textbf{(XXII)} \qquad \frac{d}{dx}(\text{ctn}^{-1} u) = -\frac{\frac{du}{dx}}{1 + u^2}.$$

In like manner we obtain the additional formulas

$$\textbf{(XXIII)} \qquad \frac{d}{dx}(\sec^{-1} u) = \frac{\frac{du}{dx}}{u\sqrt{u^2 - 1}},$$

where $0 \leqq u < \pi/2$, or $-\pi \leqq u < -\pi/2$, and

$$\textbf{(XXIV)} \qquad \frac{d}{dx}(\csc^{-1} u) = -\frac{\frac{du}{dx}}{u\sqrt{u^2 - 1}},$$

where $0 < u \leqq \pi/2$, or $-\pi < u \leqq -\pi/2$.

PROBLEMS

Differentiate each of the following functions with respect to its variable. (Nos. 1–16.)

1. $y = \tan^{-1} x^2$. *Ans.* $2x/(1 + x^4)$.
2. $y = \text{ctn}^{-1}(1/x^2)$.
3. $y = \sin^{-1} 6t$. *Ans.* $6/\sqrt{1 - 36t^2}$.
4. $y = \cos^{-1}\sqrt{1 - x^2}$.
5. $y = x \sec^{-1} x$. *Ans.* $1/\sqrt{x^2 - 1} + \sec^{-1} x$.
6. $y = x^2 \tan^{-1}(a/x)$.
7. $s = \text{ctn}^{-1}[t/(1 - t)]$. *Ans.* $-1/(1 - 2t + 2t^2)$.
8. $y = x \csc^{-1}\sqrt{1 + x^2}$.
9. $y = x^2 \sin^{-1}(2/x)$. *Ans.* $2x \sin^{-1}(2/x) - 2x/\sqrt{x^2 - 4}$.
10. $y = x\sqrt{1 - x^2} + \cos^{-1} x$.
11. $s = e^{\cos t} \cdot \sin^{-1}(1 - t)$.
    Ans. $-e^{\cos t}[1/\sqrt{2t - t^2} + \sin t \sin^{-1}(1 - t)]$.
12. $y = x \sin^{-1}\sqrt{1 - x^2} - \sqrt{1 - x^2}$.
13. $y = x \tan^{-1} ax - (1/a) \log\sqrt{1 + a^2x^2}$. *Ans.* $\tan^{-1} ax$.

14. $y = \tan^{-1}\sqrt{x^2 - 1} + \cos^{-1}(1/x)$.

15. $y = 3x \operatorname{ctn}^{-1} 3x + \log\sqrt{1 + 9x^2}$. *Ans.* $3 \operatorname{ctn}^{-1} 3x$.

16. $y = (\tan^{-1} x^2)^3$.

Find the derivative of one variable with respect to the other in the following. (Nos. 17–26.)

17. $x \sin^{-1} x + \cos 2x \sin 2y = 1$.
Ans. $dy/dx = [2 \sin 2x \sin 2y - \sin^{-1} x - x/\sqrt{1 - x^2}]/(2 \cos 2x \cos 2y)$.

18. $y\sqrt{x^2 + 2x} - \tan^{-1}\sqrt{x^2 + 2x} = 0$.

19. $y^3 \sin x + y = \tan^{-1} x$.
Ans. $dy/dx = [1/(x^2 + 1) - y^3 \cos x]/(1 + 3y^2 \sin x)$.

20. $x^2y^2 = x - \sin^{-1} 2x$.

21. $x = \sin^{-1} t,\ y = \cos^{-1} t$. *Ans.* -1.

22. $x = \sin^{-1} t^2,\ y = \tan^{-1} 2t$.

23. $\log(x^2 + y^2) - \tan^{-1}(y/x) = 0$. *Ans.* $dy/dx = (2x + y)/(x - 2y)$.

24. $\log \sin^2 y + \sec^{-1} 4x = 0$.

25. $\tan^{-1} 2x + y^2 \cdot e^{\cos x} = 7$.
Ans. $dy/dx = [y^2 \sin x \cdot e^{\cos x} - 2/(1 + 4x^2)]/(2ye^{\cos x})$.

26. $x + \sqrt{2ay - y^2} = a \cos^{-1}[(a - y)/a]$.

27. If an angle x is measured in degrees, how may you differentiate a function of it with respect to x?

28. Answer Problem 27 for $\tan^{-1} x \cdot \log \cos x$.

99. Applications of Trigonometric Differentiation. Many important problems involving trigonometric functions are included under the various topics discussed in Chapter V. In fact, many problems in that chapter may be solved readily by using trigonometric relations to express a required function.

Whenever the differentiation of trigonometric or inverse trigonometric functions is involved it is essential to remember that the formulas were derived on the assumption that the angle is expressed in radian measure. All these formulas depend on the derivative of $\sin u$, containing $\lim_{\theta \to 0} [(\sin \theta)/\theta]$ which is unity only when θ is expressed in radians.

EXAMPLES

1. A wall is to be braced by means of a beam which must pass over a lower wall b feet high and standing a feet from the first wall. What is the shortest beam that can be used?

SOLUTION. Let l be the length of the beam and let it make an angle θ with the horizontal. Then

$$l = (a + x) \sec \theta \quad \text{and} \quad x = b \operatorname{ctn} \theta.$$

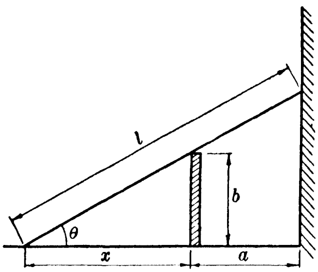


FIG. 100

Hence

$$l = a \sec \theta + b \csc \theta;$$

then

$$\frac{dl}{d\theta} = a \sec \theta \cdot \tan \theta - b \csc \theta \cdot \operatorname{ctn} \theta = 0.$$

Solving, we have

$$\frac{a \sin \theta}{\cos^2 \theta} = \frac{b \cos \theta}{\sin^2 \theta},$$

or

$$\frac{\sin^3 \theta}{\cos^3 \theta} = \frac{b}{a}, \qquad \tan \theta = \sqrt[3]{\frac{b}{a}}.$$

Applying the third test, we have

$$\frac{d^2l}{d\theta^2} = a \sec \theta \, (\sec^2 \theta + \tan^2 \theta) + b \csc \theta \, (\csc^2 \theta + \operatorname{ctn}^2 \theta).$$

Hence $d^2l/d\theta^2$ is positive and l is a minimum. The length of the beam is

$$\begin{aligned} l &= a \sqrt{1 + \left(\frac{b}{a}\right)^{2/3}} + b \sqrt{1 + \left(\frac{a}{b}\right)^{2/3}} \\ &= (a^{2/3} + b^{2/3})^{3/2}. \end{aligned}$$

2. The hands of a tower clock are $4\frac{1}{2}$ ft. and 6 ft. long, respectively. How fast are the ends approaching at four o'clock?

SOLUTION. Let s be the distance between the ends, and θ the angle between them. Then using the law of cosines, we have

$$\begin{aligned} s^2 &= 6^2 + (4.5)^2 - 12(4.5) \cos \theta \\ &= 56.25 - 54 \cos \theta. \end{aligned}$$

Hence

$$2\, s \frac{ds}{dt} = 54 \sin \theta \cdot \frac{d\theta}{dt}, \qquad \text{and} \qquad \frac{ds}{dt} = \frac{27 \sin \theta}{s} \cdot \frac{d\theta}{dt}.$$

At four o'clock $\theta = 2\,\pi/3$, $s^2 = 56.25 + 27 = 83.25$. Since θ is decreasing at the rate of $(2\,\pi - \pi/6) = 11\,\pi/6$ radians per hr., we have

$$\frac{d\theta}{dt} = -\frac{11\,\pi}{360} \text{ radians per minute.}$$

Therefore

$$\begin{aligned} \frac{ds}{dt} &= \frac{27 \dfrac{\sqrt{3}}{2} \left(-\dfrac{11\,\pi}{360}\right)}{\sqrt{83.25}} \\ &= -\,0.246 \text{ ft./min.} \end{aligned}$$

3. An acute angle A of a right triangle is computed from measurement of the sides a and b. If 1% error is assumed in each measurement, approximate the possible error in the calculated value of A.

SOLUTION. Since $\tan A = a/b$, then $A = \tan^{-1}(a/b)$ and

$$dA = \frac{b\,da - a\,db}{a^2 + b^2}.$$

To approximate the largest possible error in A, assume that $da = \pm 0.01\,a$ and $db = \mp 0.01\,b$; then,

$$dA = \pm \frac{ab}{50(a^2 + b^2)} \text{ radians.}$$

PROBLEMS

1. Find the slope of the normal to the curve $x = 2\cos^3 t$, $y = 2\sin^3 t$ at $t = \pi/6$. *Ans.* $\sqrt{3}$.

2. Find the angle between the curves $y = \cos x$ and $y = \cos 2x$ between $x = \pi/2$ and $x = \pi$.

3. Find the maximum vertical width of the cardioid $r = 1 + \cos\theta$. *Ans.* $3\sqrt{3}/2$ units.

4. Has $\lim\limits_{\theta \to 0} (\sin\theta/\theta) = 1$ any importance in the development of the calculus? Mention instances in which it is used.

5. Find the equation of the tangent and the normal to $x = a\cos\theta$, $y = b\sin\theta$ at the point determined by $\theta = \pi/4$. *Ans.* $bx + ay = ab\sqrt{2}$, $(ax - by)\sqrt{2} = a^2 - b^2$.

6. Find maximum and minimum values of $x + 2\sin x$.

7. Find θ if $\tan\theta + \operatorname{ctn}\theta$ is a minimum. *Ans.* $\pi/4$.

8. An isosceles triangle has equal sides of 6 inches which include the angle θ. If θ increases 1°/sec., how fast is the area of the triangle changing?

9. In Problem 8 find the intervals of θ in which the area of the triangle increases and decreases. *Ans.* $0 < \theta < \pi/2$, $\pi/2 < \theta < \pi$.

10. Examine $y = \sin 2x - 1$ for inflections. Evaluate maximum and minimum values for y.

11. Find maximum and minimum values of $\sin^2 x$ by calculus and check. *Ans.* 1, 0.

12. Find the slope of a cycloid at any point.

13. Show that $y = e^{-2x}\cos 2x$ and $y = e^{-2x}$ have a common tangent at any common point.

14. Derive our known formulas for $\cos 2\theta$ from $\sin 2\theta = 2\sin\theta\cos\theta$.

15. A plane is moving horizontally 200 mi./hr. at an elevation of 4200 ft. How fast will an angle of elevation from the ground change if it is near 30°? *Ans.* 1°/sec.

16. Find the minimum area cut from the first quadrant by a line through (1, 3).

17. Approximate the change in $\tan \theta$ if θ decreases $15'$ from $\pi/4$.

Ans. -0.0087 unit.

18. Find maximum and minimum values for $\log\sqrt{1+x^2} + \operatorname{ctn}^{-1} x$.

19. If $y = \sin^2 3x$, is y increasing at $x = \pi/4$; is the curve concave up at $x = \pi/6$; what is an approximate value of y if $x = 45°\ 10'$?

Ans. No; no; 0.4913.

20. Find an approximation for the change in $\cos \theta$ if θ decreases from $\pi/6$ by 0.01 radian.

21. Show that a line through any point of a rolling wheel tangent to the generated cycloid cuts the wheel at its highest point.

22. A roofer wishes to make an open gutter of maximum capacity, its cross-section being an isosceles trapezoid with lower base and equal sides 10 in. each. What should be the width of the top?

23. Approximate $\log_{10} \sin 33°$ if $\log_{10} \sin 30° = -0.301$ and $\log_{10} e = 0.434$.

Ans. -0.262.

24. A man on a wharf 20 ft. above the water pulls in the rope of a boat so that the boat approaches the wharf at the rate of 3 ft./sec., when it is 15 ft. distant. At what rate is the rope being drawn in?

25. A man slides a 10 ft. ladder up a wall, so that the foot of the ladder approaches the wall at the rate of 3 ft./sec. How fast is the top of the ladder going up the wall when the bottom is 4 ft. distant? *Ans.* $6/\sqrt{21}$ ft. sec.

26. Given $y = x^2 - \sin^{-1} x$. Find (*a*) the approximate change in x for -0.02 unit change in y; (*b*) the approximate relative and percentage errors in y for 0.02 unit error in x.

27. Evaluate the following limits.

(*a*) $\lim\limits_{\theta\to 0} \dfrac{\sin 2\theta \tan\theta}{\theta^2}$. *Ans.* 2.

(*b*) $\lim\limits_{x\to 0} \dfrac{\log(1+x+x^2) + \log(1-x+x^2)}{\sec x - \cos x}$. *Ans.* 1.

(*c*) $\lim\limits_{x\to 0} \dfrac{e^x - \cos x}{\sin x}$. *Ans.* 1.

(*d*) $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$. *Ans.* $\dfrac{1}{2}$.

28. Find a point of inflection on $y = (x+1)\tan^{-1} x$.

29. From the point $A(r, 0)$ on the circle $x^2 + y^2 = r^2$ the arc AB and the tangent AT are drawn so that $AT = AB$. Find the limiting intersections of TB with the diameter through A as B approaches A. *Ans.* $(-2r, 0)$.

30. AB is a diameter and O the center of a circle of radius r. The chord AC makes an angle θ with AB. A ray of light from A is reflected by the circle at C and the reflected ray meets AB at P. Find the limiting position of P as C approaches B. *Ans.* $OP = r/3$.

31. Solve Problem 30 if the source of light is (*a*) at M the mid-point of OA; (*b*) at N, such that A is the mid-point of ON.

Ans. (*a*) $OP = r/4$; (*b*) $OP = 2r/5$.

32. Two diametral paths cross a circular courtyard at right angles. A man walks along one path at a uniform rate, and a lamp at one extremity of the other path casts his shadow on the circular wall surrounding the court. At what rate is his shadow moving when the man is midway between the center and the extremity of the path?

33. A beam 30 feet long is to be carried horizontally around the corner of two passageways which intersect at right angles. One passageway is 10 feet wide; how narrow may the other one be to permit the beam to pass, if we assume that no allowance is necessary for its thickness? *Ans.* 11.225 ft.

100. Angle between the Radius Vector and Curve. Let the equation of the curve in polar coordinates be

$$(1) \qquad r = f(\theta),$$

and let ψ be the angle between the vector OP and the tangent PT to the curve at P (r, θ). Then give θ an increment $\Delta\theta$, and let $Q(r + \Delta r,\ \theta + \Delta\theta)$ be the corresponding point on the curve. Draw OQ, PQ, and draw PR perpendicular to OQ. Call ϕ the angle between OQ and the secant PQ. Then in the right triangles OPR and PQR,

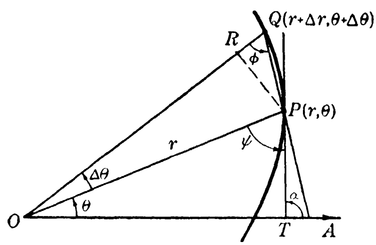


FIG. 101

$$\tan\phi = \frac{RP}{RQ} = \frac{RP}{OQ - OR}$$

$$= \frac{r \sin\Delta\theta}{r + \Delta r - r\cos\Delta\theta} = \frac{\dfrac{r\sin\Delta\theta}{\Delta\theta}}{\dfrac{\Delta r}{\Delta\theta} + \dfrac{r(1 - \cos\Delta\theta)}{\Delta\theta}}.$$

Let $\Delta\theta$ approach zero, then Q approaches P as its limit and the secant PQ approaches the tangent TP as its limit. Hence angle ϕ approaches ψ as its limit and we have

$$\lim_{\Delta\theta\to 0} \tan\phi = \tan\psi = \lim_{\Delta\theta\to 0} \frac{\dfrac{r\sin\Delta\theta}{\Delta\theta}}{\dfrac{\Delta r}{\Delta\theta} + \dfrac{r(1 - \cos\Delta\theta)}{\Delta\theta}}$$

$$= \frac{\lim\limits_{\Delta\theta\to 0} \dfrac{r\sin\Delta\theta}{\Delta\theta}}{\lim\limits_{\Delta\theta\to 0}\left[\dfrac{\Delta r}{\Delta\theta} + \dfrac{r(1 - \cos\Delta\theta)}{\Delta\theta}\right]}.$$

But, by § 55,

$$\lim_{\Delta\theta \to 0} \frac{\sin \Delta\theta}{\Delta\theta} = 1, \quad \text{and} \quad \lim_{\Delta\theta \to 0} \frac{1 - \cos \Delta\theta}{\Delta\theta} = 0,$$

therefore

$$\tan \psi = \frac{r}{\dfrac{dr}{d\theta}} = r \frac{d\theta}{dr}.$$

Hence the angle ψ between the radius vector OP and the curve $r = f(\theta)$ at P is given by

(2) $$\tan \psi = r \frac{d\theta}{dr}.$$

The inclination α of the tangent TP is given by

(3) $$\alpha = \theta + \psi.$$

To find the angle β between two curves whose equations are given in polar coordinates, the angles ψ_1 and ψ_2, which they make respectively with the radius vector to their intersection, must be found, then

(4) $$\beta = \psi_1 - \psi_2, \qquad \psi_1 \geqq \psi_2.$$

If a polar curve passes through the pole, the radius vector corresponding to the value of θ for which $r = 0$ is tangent to the curve. Hence, at the pole, $\psi = 0$. If two curves intersect at the pole, $\beta = \theta_1 - \theta_2$, where θ_1 and θ_2 are the values of θ in the respective equations for which $r = 0$.

EXAMPLES

1. Find the angle which the curve $r = 3 + 2 \cos \theta$ makes with the vector $\theta = 45°$.

SOLUTION. $$\tan \psi = \frac{r}{\dfrac{dr}{d\theta}} = \frac{3 + 2 \cos \theta}{-2 \sin \theta}.$$

Hence if $\theta = 45°$, this becomes

$$\tan \psi = \frac{3 + \sqrt{2}}{-\sqrt{2}} = -3.1213,$$

or

$$\psi = 180° - 72° \, 14.1' = 107° \, 45.9'.$$

2. What angle does the radius vector to any point of the cardioid $r = a(1 - \cos \theta)$ make with the curve?

SOLUTION.

$$\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{a(1 - \cos\theta)}{a\sin\theta} = \tan\frac{\theta}{2}.$$

Hence the angle is one-half that which the radius vector makes with the polar axis.

PROBLEMS

Find the angle of intersection of each of the following pairs of curves. (Nos. 1–14.)

1. $r = a\theta, \quad \theta = \pi/2.$ *Ans.* $57° \, 31.1'$.
2. $r = a \sin^3 (\theta/3), \quad \theta = \pi/6.$
3. $r^2 = a^2 \sin 2\theta, \quad \theta = \pi/6.$ *Ans.* $\pi/3$.
4. $r = 2 - 3\cos\theta, \quad \theta = \pi/3.$
5. $r = a(1 - \cos\theta), \quad \theta = \pi/6.$ *Ans.* $\pi/12$.
6. $r^2 = a^2 \sin (\theta/2), \quad \theta = \pi/3.$
7. $r = a[2 - (2/3)\cos\theta], \quad \theta = \pi/3.$ *Ans.* $70° \, 53.6'$.
8. $r = 2, \quad r = 2 + 3\sin\theta.$
9. $r = 1/2, \quad r = \cos\theta.$ *Ans.* $\pi/3$.
10. $r = \sin 2\theta, \quad r = \cos 2\theta.$
11. $r = 4\cos\theta, \quad r = 4\sin 2\theta.$ *Ans.* $0, \pi/2, \tan^{-1} 3\sqrt{3}$.
12. $r = a\cos\theta, \quad r = a(1 - \cos\theta)$ in the first quadrant.
13. $r = 2 + \cos\theta, \quad 4r\cos\theta = 5.$ *Ans.* $\tan^{-1}(-5/\sqrt{3}) - \pi/6$.
14. $r = a\sin\theta, \quad r^2 = a^2\cos 2\theta.$
15. Show that the spiral $r = e^{a\theta}$ cuts each radius vector at the same angle.
16. Find the inclination of $r = a(1 + \cos\theta)$ where it crosses $\theta = \pi/2$. (The angle the tangent to the curve makes with the polar axis.)
17. Find the inclination of each curve of Problem 12 at that intersection. *Ans.* $\pi/6, \pi/2$.
18. The same as Problem 17 for Problem 8.

101. Curvilinear Motion. If the path of a moving point P is a plane curve, the coordinates of P are functions of the time t, as

$$(1) \qquad x = g(t), \qquad y = f(t).$$

These are called the ***equations of motion,*** and are the parametric equations of the path.

We have already considered motion in a straight line in § 84. In curvilinear motion, the distance s measured along the arc of the

curve is also a function of the time, and the velocity, as in rectilinear motion, is $v = \lim\limits_{\Delta t \to 0} (\Delta s/\Delta t) = ds/dt$. The ***velocity*** is represented by a ***directed line-segment*** or ***vector*** whose length is the value of ds/dt. It is determined from equations (1) as follows:

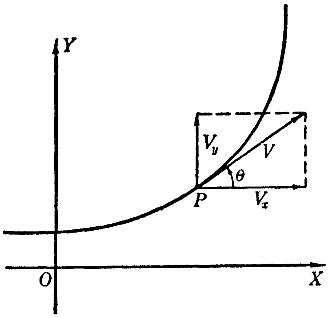


FIG. 102

In time Δt the point P is displaced a distance Δx in the direction of the x axis, and a distance Δy in the direction of the y axis. The limit of the ratios $\Delta x/\Delta t$ and $\Delta y/\Delta t$ as Δt approaches zero, namely, dx/dt and dy/dt, are the ***components of the velocity v in the directions of the coordinate axes.*** In other words, they are the projections of the vector v on lines through P parallel to the coordinate axes. These vector components are designated v_x and v_y respectively. Let v make an angle θ with v_x, then

$$(2) \qquad v_x = \frac{dx}{dt} = v \cos \theta, \qquad v_y = \frac{dy}{dt} = v \sin \theta.$$

From these we have

$$(3) \qquad v = \pm \sqrt{v_x^2 + v_y^2}, \qquad \tan \theta = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{dy}{dx},$$

which give the magnitude and direction of the velocity. Hence the direction of the velocity at any instant is along the tangent to the curve at P.

Writing the positive value of v in the form

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2},$$

we have at once an expression for the differential of arc ds since s, x, and y are functions of t, namely,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

whence

(4) $$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

EXAMPLE

A particle is moving on the curve $x = a\theta - b \sin \theta$, $y = a - b \cos \theta$. Find the rectangular equation of its path, the velocity in its path, and the point where the direction of motion has the greatest inclination.

SOLUTION. The equation of the path is found by eliminating the parameter θ. From the second equation,

$$\theta = \cos^{-1} \frac{a - y}{b}.$$

Substituting this in the first equation, we get

$$x = a \cos^{-1} \frac{a - y}{b} - \sqrt{b^2 - (a - y)^2}.$$

Since x, y, and θ must be functions of t,

$$v_x = \frac{dx}{dt} = (a - b \cos \theta) \frac{d\theta}{dt},$$

$$v_y = \frac{dy}{dt} = b \sin \theta \frac{d\theta}{dt}.$$

Hence

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{a^2 + b^2 - 2\, ab \cos \theta}\, \frac{d\theta}{dt}.$$

That inclination will be greatest which has the greatest absolute value for its slope. The slope is

$$y' = \frac{dy}{dx} = \frac{v_y}{v_x} = \frac{b \sin \theta}{a - b \cos \theta}.$$

$$\frac{dy'}{d\theta} = \frac{ab \cos \theta - b^2 \cos^2 \theta - b^2 \sin^2 \theta}{(a - b \cos \theta)^2} = \frac{b(a \cos \theta - b)}{(a - b \cos \theta)^2}.$$

Then

$$\frac{dy'}{d\theta} = 0 \text{ when } \theta = \cos^{-1} \frac{b}{a},$$

$$= \infty \text{ when } \theta = \cos^{-1} \frac{a}{b}.$$

If $b < a$, $\theta = \cos^{-1} (b/a)$ gives the maximum slope.
If $a < b$, the direction of motion is vertical when $\theta = \cos^{-1} (a/b)$.

102. Acceleration in Curvilinear Motion. Let the change in velocity along the x axis in time Δt be Δv_x and the corresponding

change along the y axis be Δv_y. Then

$$\lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = g''(t) = a_x,$$

and

$$\lim_{\Delta t \to 0} \frac{\Delta v_y}{\Delta t} = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = f''(t) = a_y,$$

are the components of the acceleration in the direction of the coordinate axes, respectively. These are likewise represented by vectors, and the total acceleration a is

$$(1) \qquad a = \sqrt{[g''(t)]^2 + [f''(t)]^2} = \sqrt{a_x^2 + a_y^2}.$$

It is very important to observe here that *the acceleration of a moving point in curvilinear motion is not given by the value dv/dt* as is the case in rectilinear motion.

The velocity is directed along the tangent to the curve, and dv/dt is the component of the acceleration in the direction of motion. It is called the ***tangential acceleration*** a_T.

The component of the acceleration at right angles to the direction of motion is called the ***normal acceleration*** a_N.

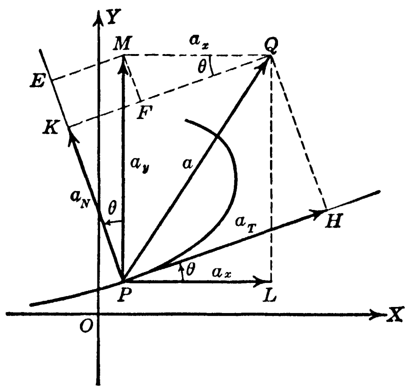


FIG. 103

Differentiating $v = \sqrt{v_x^2 + v_y^2}$ with respect to t, we have

$$(2) \qquad a_T = \frac{dv}{dt} = \frac{v_x a_x + v_y a_y}{v} = \frac{x'x'' + y'y''}{(x'^2 + y'^2)^{1/2}}.$$

Let the vectors PL and PM represent the horizontal and vertical components of the acceleration of a point P moving along a given curve. Then the vector PQ represents the total acceleration. The projections of PQ on the tangent and normal at P are PH, the tangential acceleration, and PK, the normal acceleration. Let the angle between the vectors a_x and a_T be θ; then

$$PH = KQ = EM + FQ,$$

or

$$a_T = a_y \sin \theta + a_x \cos \theta.$$

Substituting for $\sin\theta$ and $\cos\theta$ the values from (**2**), § 101, we obtain the result given above in (**2**).

Similarly, to find the normal acceleration, we have

$$PK = PE - FM = a_y \cos\theta - a_x \sin\theta,$$

and substituting as before, we obtain

$$(3) \qquad a_N = \frac{v_x a_y - v_y a_x}{v} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{1/2}},$$

where the primes, as in (**2**), indicate differentiation with respect to t.

The normal component of the acceleration produces no effect on the speed of the moving point but it is the ***centrifugal force*** which causes the change in the direction of motion.

PROBLEMS

Find the equation of the path in rectangular coordinates and the velocity of a point along the path if its equations of motion are as stated below. (Nos. 1–9.)

1. $x = 2t, \quad y = 2/(t^2 + 1), \quad t = 1.$
Ans. $y(x^2 + 4) = 8$, $\sqrt{5}$ units per unit time.

2. $x = 2 - 3\cos t, \quad y = 3 + 2\sin t.$ Also find a.

3. $x = \pm\sqrt{t^2 + 9}, \quad y = t.$ *Ans.* $x^2 - y^2 = 9$, $\sqrt{(2t^2 + 9)/(t^2 + 9)}$.

4. $x = 2t, \quad y = 2\sqrt{t - t^2}.$ Also find a.

5. $x = 4\cos^3 2t, \quad y = 4\sin^3 2t.$ Find the smallest positive value of t for which v is a maximum. *Ans.* $x^{2/3} + y^{2/3} = 4^{2/3}$; $12\sin 4t$; $t = \pi/8$.

6. $x = \log t, \quad y = 2\sqrt{t}.$ Also find a.

7. $x = e^{2t}, \quad y = \log e^{-t}.$ *Ans.* $x = e^{-2y}$, $\sqrt{4e^{4t} + 1}$.

8. $x = at - b\sin t, \quad y = a - b\cos t.$ Also find a.

9. $x = e^{-2t}\cos 2t, \quad y = e^{-2t}\sin 2t.$
Ans. $x = \sqrt{x^2 + y^2}\cos^{-1}[\log(x^2 + y^2)^{-1/2}]$, $2e^{-2t}\sqrt{2}$.

10. If $v_x = 5$ ft./sec. on $x^2 + y^2 = 10$, find v at $x = 2$.

11. If $v_x = 15$ ft./sec. along $x^2 - y^2 = 144$, find v at $(13, 5)$.
Ans. 41.8 ft./sec.

12. A point moves along $x^2 - y^2 = 25$. Where is $v_x = v_y$?

13. A body moves along $y^2 - x^2 + 4 = 0$ with $v_x = 4$ units/sec. Find v_y and a_y. *Ans.* $4x/y$, $-64/y^3$.

14. A body moves along $x^2 - 3y^2 = 6$ with $v_x = 2$ units/sec. Find v at $(3, 1)$.

15. A particle moves along $y = \cos x$ at 1 unit/sec. Find v_x, v_y at $x = \pi/6$. *Ans.* $2/\sqrt{5}$ units/sec.; $-1/\sqrt{5}$ units/sec.

16. Motion along $y = 2\sin(2x - 3)$ has $v_x = 5$ units/sec. Where is the motion fastest; slowest?

17. A projectile followed the curve whose equation may be taken as $y = x - (x^2/100)$, where x and y are measured in feet. If the horizontal velocity was 20 ft./sec., how fast was the projectile rising at $x = 0$? What was its velocity when $x = 100$ ft.? *Ans.* 20 ft./sec., $20\sqrt{2}$ ft./sec.

18. A particle moves along $y = \sin x$. Find a point where $v_x = 2v_y$.

19. The laws of motion of a particle in a plane are $x = a\cos t + b$, $y = a\sin t + c$. Find the rectangular equation of its path and show that its velocity is constant. *Ans.* $(x - b)^2 + (y - c)^2 = a^2$.

20. A man walks along the 200 ft. diameter of a semicircular courtyard at 5 ft./sec. How fast is his shadow moving along the wall if the sun's rays are perpendicular to the diameter?

103. Angular Velocity. If a wheel turns, the angle which is generated is dependent upon the time and hence is a function of the time. The rate of change of θ with respect to the time is called the ***angular velocity*** and is generally denoted by ω. That is,

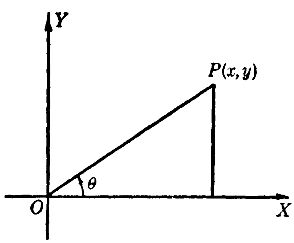


Fig. 104

$$(1) \qquad \omega = \frac{d\theta}{dt}.$$

Likewise the ***angular acceleration*** is

$$(2) \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

For a point $P(x, y)$ we can find the expression for the ω of OP in terms of x and y as follows:

$$\theta = \tan^{-1}\frac{y}{x}.$$

Differentiating with respect to t, we get

$$\omega = \frac{d\theta}{dt} = \frac{1}{1 + (y/x)^2} \cdot \frac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2},$$

or

$$(3) \qquad \omega = \frac{xv_y - yv_x}{x^2 + y^2}.$$

Hence ω depends on the position of P and on its horizontal and vertical component velocities.

If the point P is at a fixed distance from O, we have the special case of the angular velocity of a point on a circle. Then the distance traversed by P is a function of θ, namely $r\theta$, and its velocity is

$$v = \frac{ds}{dt} = r \cdot \frac{d\theta}{dt} = r\omega.$$

Therefore $\omega = v/r$, where v is the linear velocity of the point on the circle. Also

$$\alpha = \frac{d\omega}{dt} = \frac{1}{r}\frac{dv}{dt} = \frac{a}{r},$$

where a denotes the linear acceleration of the point along the circumference of the circle.

Angular velocity is measured in radians per unit of time, or in revolutions per unit of time.

EXAMPLES

1. A point on the rim of a flywheel of radius 5 ft., which is 4 ft. above the level of the center of the wheel, has a vertical velocity of 50 ft./sec. Find the angular velocity of the wheel.

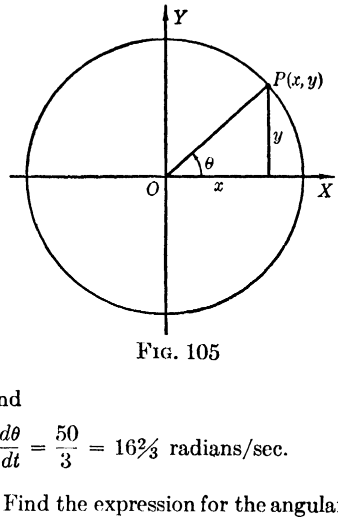


FIG. 105

SOLUTION. The point P is located by the equations

$$x = 5\cos\theta, \qquad y = 5\sin\theta.$$

Since v_y is given, we have

$$v_y = \frac{dy}{dt} = 5\cos\theta \cdot \frac{d\theta}{dt} = 50.$$

When the ordinate of P is 4, $\cos\theta = 3/5$ and

$$\frac{d\theta}{dt} = \frac{50}{3} = 16\tfrac{2}{3} \text{ radians/sec.}$$

2. Find the expression for the angular velocity of the line joining the origin to a point on the line $2x - 3y = 4$. Evaluate this when the point is at (5, 2) on the line, and is moving along it at the rate of 5 ft./sec.

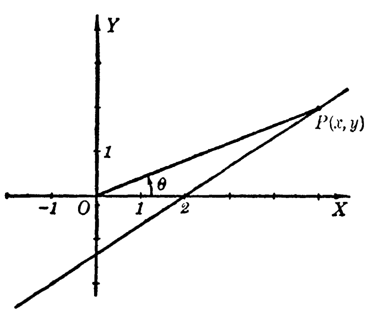


FIG. 106

SOLUTION. $\theta = \tan^{-1}\dfrac{y}{x}$,

then

$$\omega = \frac{xv_y - yv_x}{x^2 + y^2}$$

and $2\,v_x - 3\,v_y = 0$, or $v_x = (3/2)v_y$; also $x = (3\,y + 4)/2$. Hence

$$\omega = \frac{\left(2 + \frac{3\,y}{2}\right) v_y - y\,\frac{3\,v_y}{2}}{\left(2 + \frac{3\,y}{2}\right)^2 + y^2} = \frac{2\,v_y}{4 + 6\,y + \frac{13\,y^2}{4}}.$$

Now $v = \sqrt{v_x^2 + v_y^2} = \sqrt{13\,v_y^2/4}$ (§ 101), and for $v = 5$ ft./sec.,

$$\frac{13\,v_y^2}{4} = 25, \qquad v_y = \pm\frac{10}{\sqrt{13}};$$

then at the point (5, 2),

$$\omega = \pm\frac{20}{29\sqrt{13}} = \pm\,0.1913 \text{ radian/sec.}$$

The double sign permits motion up or down the line.

PROBLEMS

1. A wheel 4 feet in diameter revolves on a fixed axis at the rate of 20 r.p.m. Find v_x of a point of the rim 1 ft. above the center.

Ans. $\pm\,2\,\pi/3$ ft./sec.

2. In Problem 1, find v_x and v_y of a point on the rim 1 ft. below the axis.

3. For the wheel of Problem 1 suppose $v_x = 2$ ft./sec. at a point on the rim 1 ft. above the axis and to the left of the center. Find the direction the wheel revolves and its ω in r.p.m. *Ans.* Clockwise, $60/\pi$ r.p.m.

4. A wheel 10 feet in diameter makes 3 r.p.m. on its fixed axis. What are ω, v_x, v_y for a point on the rim 3 ft. above the axis?

5. A flywheel of fixed center revolves in a counter-clockwise direction 20 r.p.m. If its radius is 5 feet, at what points of the rim is v_x equal to $-\,160\,\pi$ ft./min.? *Ans.* 3 ft. to left or right of center.

6. A line joins the origin and a point of $x + 3\,y = 4$. Find the ω of this line in terms of x and v_x as the point moves along $x + 3\,y = 4$. Evaluate when the point is at $(7, -1)$ and the motion is 3 units/sec.

7. A point moves along the parabola $r = 6/(1 - \sin\theta)$ so that the angular velocity of the radius vector to the point is 1.25 radians per second. If r is measured in inches, how fast is r changing at $\theta = 30°$? *Ans.* $15\sqrt{3}$ in./sec.

8. A point moves along the hyperbola $r = 4/(1 + 2\cos\theta)$ so that r is increasing 3 in./sec. Find ω of the radius vector at $\theta = \pi/3$.

9. A point P moves along the parabola $4\,y^2 = 5\,x$ so that its ordinate increases 2 units per second. Find ω of the line joining P to the origin.

Ans. $-\,8/(4\,x + 5)$.

10. A hoop of radius 2 ft. rolls so that ω of a radius is 4 radians per second. Find v_x and v_y of a point on the rim.

11. A wheel 4 ft. in diameter rolls along a straight horizontal road at 20 r.p.m. Find v and a, and the direction of motion of a point on the rim when it is 1 ft. below the center of the wheel and rising.

Ans. $4\,\pi/3$ ft./sec.; $8\,\pi^2/9$ ft./sec.^{2}; $\pi/3$.

12. A 6-foot wheel rolls along a horizontal straight road 30 mi./hr. What are v_x, v_y of a point on the rim? Evaluate at high and low and mid-point.

13. A variable point P on a rod OA which rotates with uniform angular velocity about O describes the curve whose equation, with O as the pole, is $r = a(1 + \cos \theta)$. What is the velocity of P at any time?

Ans. $k\sqrt{2\,ar}$.

14. A rod OA revolves in a vertical plane with uniform ω about O. A movable point P on the rod is constrained by a cam to describe $r = a(2 - \cos \theta)$, where $OP = r$, and θ is the angle between OP and the horizontal. Find the component of v along the rod, and the total v.

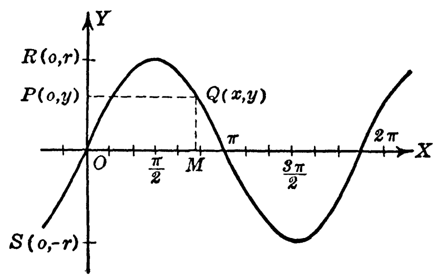
104. Simple Harmonic Motion. If a point P oscillates in a straight line through a fixed point O as center, so that its acceleration is always proportional to the directed distance from P to O, then P is said to describe ***simple harmonic motion.***

Let O be an origin and let the distance of P from O be s. By the definition, the acceleration of P must satisfy the relation

$$a = \frac{d^2s}{dt^2} = -\,k^2 s. \tag{1}$$

While the general solution of (1) for s is obtained in a later chapter, it is obvious that the equation is satisfied if s is proportional to either $\sin kt$ or $\cos kt$.

To illustrate this motion, let a point Q move along the curve $y = r \sin x$ so that the horizontal components of its velocity is a constant k. Then, if $t = 0$ when $x = 0$, the coordinates of Q are:



$$x = kt, \quad y = r \sin kt.$$

FIG. 107

Let the projection of Q on the y axis be $P(0, y)$. Then, since $d^2y/dt^2 = -\,k^2 r \sin kt = -\,k^2 y$, it is evident that P describes simple harmonic motion as it oscillates between R and S. If the time is t_0 when $x = 0$, then $OM = k(t - t_0)$ and

$$y = r \sin k(t - t_0) = r \sin (kt - \beta), \tag{2}$$

where $\beta = kt_0$.

The period of P is the time of one complete oscillation and is the difference of the two values of t when the angle $kt - \beta$ varies

from 0 to 2π. Hence the period of t is $2\pi/k$. When P is at R or at S, OP has a maximum absolute value r, which is called the ***amplitude*** of the motion. The ***phase angle*** is β or kt_0, and t_0 is the phase expressed in time.

Another illustration is as follows: Let Q move with constant speed v in a circular path. Then its projection P upon any line in the plane of its path describes simple harmonic motion. Take a diameter as the line and call it the x axis. If the time is t_0 when the point is at R, and t when at Q, then $\theta = k(t - t_0)$, where the constant k is v/r, the angular velocity. Then for any position of P

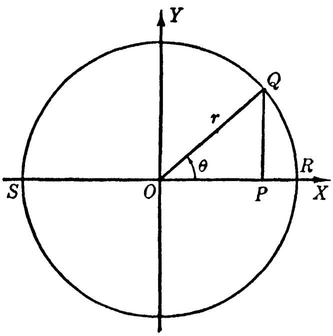


FIG. 108

(3) $$x = r\cos k\,(t - t_0) = r\cos\,(kt - \beta).$$

The ***velocity*** of the point P is

(4) $$\frac{dx}{dt} = -kr\sin\,(kt - \beta),$$

and its ***acceleration***

(5) $$\frac{d^2x}{dt^2} = -k^2r\cos\,(kt - \beta) = -k^2x.$$

When P is at R or at S, the distance x has an extreme value and dx/dt is zero; that is, when $\sin\,(kt - \beta) = 0$ or $t = (n\pi + \beta)/k$, where n is an integer. The velocity of P has an extreme value when the acceleration is zero, that is, when P is at O, or when $\cos\,(kt - \beta) = 0$. This makes $t = [(2n + 1)\pi + 2\beta]/2k$, where n is any integer.

Equations (2) and (3) may be written in the form

(6) $$s = A\sin kt + B\cos kt,$$

where s represents the distance in each formula.

The importance of simple harmonic motion is due to its representation of the motion of vibrating bodies such as that of a weight suspended by a spring, or a vibrating string or wire. Since the

acceleration of any particle is proportional to the force actin·; on it, we have

$$F = \rho m \frac{d^2s}{dt^2} = -\rho mrk^2 \cos(kt - \beta) = -\rho mk^2 s.$$

Hence the force which tends to restore the vibrating particle to its central position is directly proportional to its distance from that central position and is always directed toward the center.

EXAMPLE

1. Write the law of motion for a particle which moves with simple harmonic motion of period 5 seconds and amplitude 4 feet.

SOLUTION. Take the motion along the x axis. Then

$$x = r \cos(kt - \beta),$$

where r, the amplitude, is 4, and the period $2\pi/k = 5$. Hence $k = 2\pi/5$. Then

$$x = 4 \cos\left(\frac{2\pi t}{5} - \beta\right).$$

PROBLEMS

1. Find the velocity, the acceleration, the period, and the amplitude of the particle which moves according to the law $s = 2c \sin(kt + \theta)$.

Ans. $2ck \cos(kt + \theta)$; $-k^2 s$; $2\pi/k$; $2c$.

Discuss the motion defined by each of the following equations. (Nos. 2–8.) Compare the acceleration and the distance.

2. $x = \cos 3t$.

3. $s = 2 \sin(t/2 + \pi/4)$.

Ans. $\cos(t/2 + \pi/4)$; $-s/4$; 4π; 2; S. H. M.

4. $s = a \cos bt + c \sin bt$.

5. $x = \sin t - (1/2) \cos 2t$.

Ans. $\cos t + \sin 2t$; $-\sin t + 2\cos 2t$; not S. H. M.

6. $s = \sin t \cos t - 2 \cos 2t$.

7. $s = 2 - 4 \sin^2 t + (3/2) \sin 2t$.

Ans. $3 \cos 2t - 4 \sin 2t$; $-4s$; π; $5/2$.

8. $s = (1/2) \cos 2t - (1/3) \cos 3t$.

9. An alternating electric current varies in intensity according to the law $c = a \sin kt$. What is the maximum current, and what is the frequency?

Ans. a; π/k is time between $c = a$, $c = -a$.

10. A particle moves according to the law $s = c_1 e^{kt} + c_2 e^{-kt}$. Find its velocity and its acceleration. Is this motion simple harmonic? Explain.

11. A flywheel 5 ft. in diameter revolves with a uniform velocity of 40 revolutions per min. What law defines the motion of the projection of a point on the rim on a plane parallel to the shaft of the wheel? Discuss.
Ans. $s = (5/2) \cos (4\pi t/3 + k)$; $a = -16\pi^2 s/9$; $p = 1\frac{1}{2}$ sec.; amp. $= 5/2$.

12. If $s = 3 - 6 \sin^2 2t$ expresses the motion of a body in a straight line show that the motion is simple harmonic and discuss it.

13. If a particle moves in simple harmonic motion with $v = 3$ ft./sec. when $s = 3$ ft. and $v = 4$ ft./sec. when $s = 2$ ft., find the law of motion and discuss it. *Ans.* $s = 6\sqrt{3/7} \sin t\sqrt{7/5}$; $p = 2\pi\sqrt{5/7}$; amp. $= 6\sqrt{3/7}$.

14. The amplitude of a simple harmonic motion is 6 ft., and when the body is midway between the mean and extreme positions, the numerical value of its velocity is 3 ft./sec. Find the period and discuss the motion.

105. Curvature. In a given circle of radius R the length of arc Δs between two points P and Q is proportional to the angle $\Delta\alpha$ between their tangents; that is,

$$\Delta s = R \cdot \Delta\alpha,$$

where $\Delta\alpha$ is expressed in radians. Then

$$R = \frac{\Delta s}{\Delta\alpha}$$

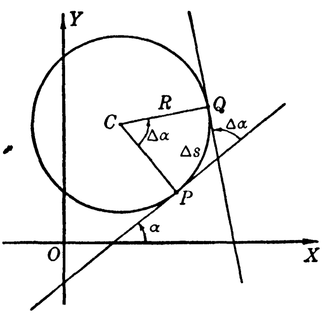


FIG. 109

is the constant rate of change of the length of arc with respect to its inclination. Since s is measured in linear units and α in radians, R is the rate of change of s per radian. The reciprocal ratio,

$$\frac{\Delta\alpha}{\Delta s} = \frac{1}{R},$$

which is the rate of change of α per unit length of arc, is called the ***curvature*** of the circle. Hence a given circle of radius R has a curvature $1/R$.

If, however, we consider any other curve, the rate of change of the length of arc with respect to its inclination is in general a variable quantity. Let P and Q be any two points on a given curve separated by a length of arc Δs. Let the inclinations of the tangents at P and Q be α and $\alpha + \Delta\alpha$ respectively. The ratio $\Delta\alpha/\Delta s$ is the ***average rate of change of α with respect to the arc s.***

It is called the ***average curvature*** of the arc PQ. Now let Q approach P; then

(1) $$\lim_{\Delta s \to 0} \frac{\Delta \alpha}{\Delta s} = \frac{d\alpha}{ds} = K,$$

is defined as the ***curvature at P***. That is, *the curvature of a curve at a given point is the rate of change of its inclination with respect to its length of arc.*

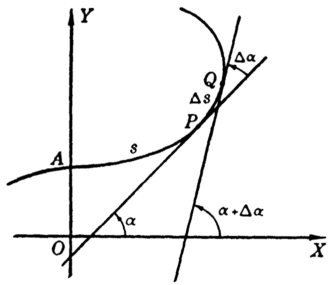


FIG. 110

106. Formulas for Curvature. Let $y = f(x)$ be the equation of a given curve. Since the slope of the curve at P is

(1) $$y' = \tan \alpha$$

the inclination is therefore

(2) $$\alpha = \tan^{-1} y'.$$

Then

(3) $$\frac{d\alpha}{ds} = \frac{d\alpha}{dx} \cdot \frac{dx}{ds} = \frac{y''}{1 + y'^2} \cdot \frac{dx}{ds}.$$

But, from the differential of arc ds in (4) § 101, we have

$$\frac{ds}{dx} = \sqrt{1 + y'^2}.$$

Substituting this in (3), we have

(4) $$K = \frac{d\alpha}{ds} = \frac{y''}{(1 + y'^2)^{3/2}}.$$

If y is considered the independent variable in the equation, we may write

$$\tan \alpha = \frac{1}{\frac{dx}{dy}}, \qquad \alpha = \tan^{-1} \frac{1}{\frac{dx}{dy}}.$$

Differentiating with respect to y and using the relation

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2},$$

we obtain

$$(5) \qquad K = -\frac{x''}{(1 + x'^2)^{3/2}},$$

where the primes indicate differentiation with respect to y.

If the curve is defined by the parametric equations

$$x = g(t), \qquad y = f(t),$$

we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad \frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} - \frac{dy}{dt} \cdot \frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}.$$

Substituting these in (4), we obtain

$$(6) \qquad K = \pm \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}},$$

where the sign is that of x', and where the primes indicate differentiation with respect to t. If we take the sign of the radical in (4) as positive, then the sign of K will depend on that of y'', that is, it will be *positive or negative according as the curve is concave upward or concave downward.*

EXAMPLE

Find the curvature of the parabola $x^{1/2} + y^{1/2} = a^{1/2}$, at any point (x, y) of the curve.

SOLUTION. Differentiating implicitly, we obtain

$$y' = -\sqrt{\frac{y}{x}},$$

$$y'' = \frac{\frac{-y'\sqrt{x}}{\sqrt{y}} + \frac{\sqrt{y}}{\sqrt{x}}}{2x} = \frac{1 + \sqrt{\frac{y}{x}}}{2x} = \frac{a^{1/2}}{2x^{3/2}},$$

then

$$K = \frac{\frac{a^{1/2}}{2x^{3/2}}}{\left(1 + \frac{y}{x}\right)^{3/2}} = \frac{a^{1/2}}{2(x + y)^{3/2}}.$$

107. Curvature in Polar Coordinates. If the equation of the curve given in polar coordinates is

$$(1) \qquad r = f(\theta),$$

then

$$(2) \qquad K = \frac{d\alpha}{ds} = \frac{d\alpha}{d\theta} \cdot \frac{d\theta}{ds}.$$

Now $\alpha = \theta + \psi$ and, by § 100,

$$\tan \psi = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{r'}, \qquad \psi = \tan^{-1}\frac{r}{r'}.$$

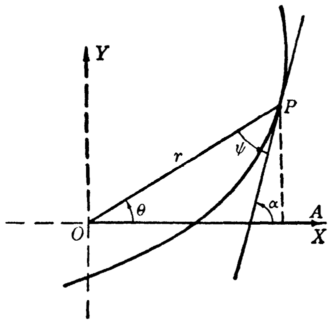


FIG. 111

Then

$$(3) \qquad \frac{d\alpha}{d\theta} = 1 + \frac{r'^2 - rr''}{r^2 + r'^2} = \frac{r^2 + 2\,r'^2 - rr''}{r^2 + r'^2}.$$

To express ds in polar coordinates, use the transformation

$$x = r\cos\theta, \qquad y = r\sin\theta \quad (\S\ 18,\ \text{VIII}),$$
$$dx = \cos\theta\,dr - r\sin\theta\,d\theta, \qquad dy = \sin\theta\,dr + r\cos\theta\,d\theta,$$

then

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\theta^2},$$

or

$$(4) \qquad \frac{ds}{d\theta} = \sqrt{r^2 + r'^2}.$$

Substituting (3) and (4) in (2), we get

$$(5) \qquad K = \frac{r^2 + 2\,r'^2 - rr''}{(r^2 + r'^2)^{3/2}},$$

the independent variable being θ.

EXAMPLE

Find the curvature of the cardioid $r = a(1 - \cos\theta)$ at any point.

SOLUTION. Here $r' = a\sin\theta$, $r'' = a\cos\theta$.

Substituting in the formula, we obtain

$$K = \frac{a^2[(1 - \cos\theta)^2 + 2\sin^2\theta - \cos\theta(1 - \cos\theta)]}{a^3[(1 - \cos\theta)^2 + \sin^2\theta]^{3/2}}$$
$$= \frac{3(1 - \cos\theta)}{a[2(1 - \cos\theta)]^{3/2}} = \frac{3}{2\,a\sqrt{2(1 - \cos\theta)}} = \frac{3}{2\sqrt{2\,ar}}.$$

PROBLEMS

Find the curvature of each of the following curves at any point and calculate the numerical value when definite points are indicated.

1. $y = xe^{-x}$ at $x = 1$. *Ans.* $-1/e$.
2. $9x^2 + 4y^2 = 36$ at $(2, 0)$.
3. $x^2/9 - y^2/4 = 1$ at $(3, 0)$. *Ans.* $3/4$.
4. $y = \log \sec x$.
5. $xy^2 = a^2(a - x)$ at $x = a$. *Ans.* $\pm 2/a$.
6. $y = x^3(x - 2)^2$ at $(2, 0)$.
7. $y = e^x$ at the point where the curvature is a maximum.
Ans. $(-\log\sqrt{2}, \sqrt{1/2})$, $K = 2/(3\sqrt{3})$.
8. $\sqrt{x} - \sqrt{y} = \sqrt{a}$.
9. $y = e^{-2x}\sin 3x$ at the origin. *Ans.* $-6/(5\sqrt{10})$.
10. $y = (x^2 \log x)/e$ at $x = e$.
11. $x = t + t^2$, $y = t - t^2$ at $t = 1$. *Ans.* $-2/(5\sqrt{10})$.
12. $x = a\cos\theta$, $y = b\sin\theta$ at $\theta = \pi/2$.
13. $x = 2\cos\phi$, $y = \sin^2\phi$ at $\phi = \pi$. *Ans.* $-1/(4\sqrt{2})$.
14. $x = t^2$, $y = 2t^3e^t$ at $t = 1$.
15. $x = \cos t$, $y = t\sin t$ at $t = \pi/3$.
Ans. $-18(4\pi - 3\sqrt{3})/(\pi^2 + 6\pi\sqrt{3} + 54)^{3/2}$.
16. $x = \log\sin 2\theta$, $y = \log\cos 2\theta$ at $\theta = \pi/8$.
17. $x = e^t\cos t$, $y = e^t\sin t$ at $t = \pi/2$. *Ans.* $-1/(e^{\pi/2}\sqrt{2})$.
18. $x = e^{-2t}\cos 2t$, $y = e^{-2t}\sin 2t$.
19. $r = e\theta$. *Ans.* $(\theta^2 + 2)/e(\theta^2 + 1)^{3/2}$.
20. $r^2 = a^2\sin 2\theta$.
21. $r = a(1 + \cos\theta)$ at $\theta = \pi/4$. *Ans.* $3/(2a\sqrt{2 + \sqrt{2}})$.
22. $r = a\sec^2(\theta/2)$.

108. Circle of Curvature. Center of Curvature. Let P be any point on the curve $y = f(x)$. *The circle which is tangent to the curve at P on the concave side, and which has the same curvature as the curve at P, is called the* ***circle of curvature*** *at P.*

Since the circle is tangent to the curve at P, its center must lie on the normal at P. The curvature of the circle is $K = 1/R$. Hence to construct the circle draw the normal at P, and on the concave side of the curve mark a point C such that $PC = R = 1/K$. With this point as center, describe a circle with radius R.

The center C of this circle is called the ***center of curvature*** at P, and the radius R is called the ***radius of curvature*** at P. Hence, at any point of the curve, the radius of curvature is the reciprocal of the curvature, that is,

$$(1) \qquad R = \frac{1}{K} = \frac{(1 + y'^2)^{3/2}}{y''}.$$

To find the coordinates (h, k) of C we use the relations, distance $PC = R$ and slope $PC = -1/y'$. These give

$$(2) \qquad R^2 = (h - x)^2 + (k - y)^2 = \frac{(1 + y'^2)^3}{y''^2},$$

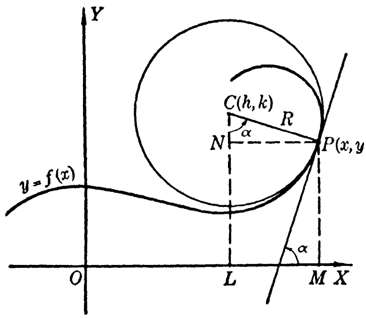


FIG. 112

and

$$(3) \qquad (h - x) + y'(k - y) = 0.$$

Eliminating $(h - x)$ between these, we have

$$(4) \qquad k - y = \pm \frac{1 + y'^2}{y''},$$

where the positive sign in (4) must be used, since $(k - y)$ will have the same sign as y'', as C is on the concave side of the curve. Then

$$(5) \qquad h = x - \frac{y'(1 + y'^2)}{y''}, \qquad k = y + \frac{1 + y'^2}{y''}.$$

109. Osculating Circle. *The* ***osculating circle*** *of a given curve at a point P is the limiting position of the circle through P and two other points P_2 and P_4 of the curve as P_2 and P_4 approach P.*

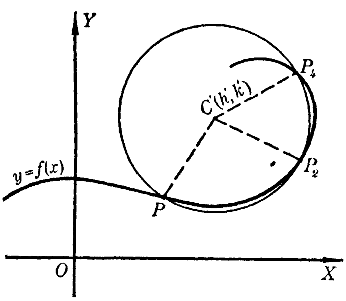


FIG. 113

Let $y = f(x)$ be the equation of a given curve and let the circle through the points P, P_2, P_4 on the curve have center at $C'(h', k')$ and radius R'. Its equation is

$$(1) \qquad (x - h')^2 + (y - k')^2 - R'^2 = 0.$$

Since P, P_2, and P_4 are on both the curve and the circle, if we

denote by $\phi(x)$ the left-hand member of (1) when y is replaced by $f(x)$, then $\phi(x)$ will vanish for the abscissas of P, P_2, and P_4, that is,

$$(2) \qquad \begin{cases} \phi(x) = 0, \\ \phi(x_2) = 0, \\ \phi(x_4) = 0. \end{cases}$$

Hence, by Rolle's Theorem, $\phi'(x)$ must vanish for at least two values of x, one at P_1 for x_1 between x and x_2, the other at P_3 for x_3 between x_2 and x_4, that is,

$$(3) \qquad \phi'(x_1) = 0, \quad \phi'(x_3) = 0.$$

Applying the same theorem again, $\phi''(x)$ must vanish for some value of the variable between x_1 and x_3, say x_0, then

$$(4) \qquad \phi''(x_0) = 0.$$

Now let P_2 and P_4 approach P; then P_0, P_1, P_3 will each approach P as a limit, and h', k', and R' will approach h, k, and R of the osculating circle at P. Then, corresponding to (2), (3), and (4), we have the limiting values of ϕ, ϕ', and ϕ'', namely,

$$(5) \qquad \phi(x) = (x - h)^2 + (y - k)^2 - R^2 = 0,$$

$$(6) \qquad \phi'(x) = 2[(x - h) + (y - k)y'] = 0,$$

$$(7) \qquad \phi''(x) = 2[1 + y'^2 + (y - k)y''] = 0,$$

respectively.

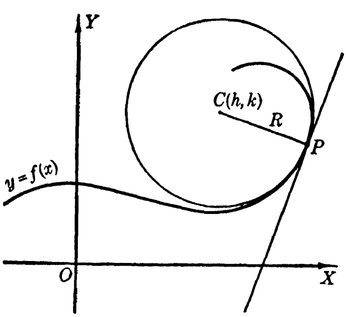


FIG. 114

These relations enable us to find h, k, and R in terms of x, y, y' and y''. Solving these, we have

$$(8) \qquad \begin{cases} h = x - \dfrac{y'(1 + y'^2)}{y''}, \\ k = y + \dfrac{1 + y'^2}{y''}, \\ R = \dfrac{(1 + y'^2)^{3/2}}{y''}. \end{cases}$$

But this is just the circle of curvature at P.

Hence, *the circle of curvature at any point of a curve is the osculating circle at that point.*

This means that the circle of curvature has three coincident intersections with the curve; hence it will in general cross the curve at P. This circle "fits" the curve more closely at P than any other circle that can be drawn. To show the direction of the curve at P, we draw the tangent to the circle, since both have the same slope y'. In exactly the same way, the circle of curvature shows the curvature at P, since both circle and curve are readily shown to have the same value of y''.

110. Limiting Position of Intersection of Adjacent Normals. Let P and P_1 be two points on the curve $y = f(x)$. At each point draw the normal to the curve; the slopes will be $-1/y'$ and $-1/y_1'$, respectively.

Let (h', k') be the intersection of these normals; then

$$(1) \qquad \begin{cases} (x - h') + y'(y - k') = 0, \\ (x_1 - h') + y_1'(y_1 - k') = 0. \end{cases}$$

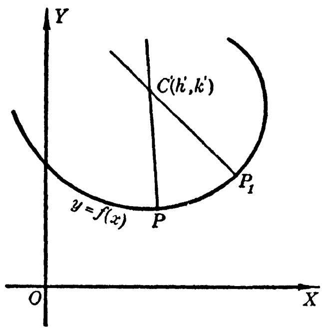


FIG. 115

Let

$$(x - h') + y'(y - k') = \psi(x),$$

where $y = f(x)$. Then, by (1),

$$(2) \qquad \psi(x) = 0, \quad \psi(x_1) = 0.$$

Hence, by Rolle's Theorem, we have

$$(3) \qquad \psi'(x_0) = 0,$$

where x_0 is some value of the variable between x and x_1. Now let P_1 approach P along the curve; then P_0 will approach P as a limit and the point (h', k') will approach some limiting intersection, say (h, k). The limiting values of $\psi(x)$ and $\psi'(x)$ will be one-half of $\phi'(x)$ and $\phi''(x)$, respectively, in (6) and (7) of the preceding article. That is, h and k must satisfy the relations

$$(4) \qquad \psi(x) = (x - h) + y'(y - k') = 0,$$

$$(5) \qquad \psi'(x) = 1 + y'^2 + (y - k)y'' = 0.$$

Solving these for h and k, we get the center of curvature at P. Therefore, *the center of curvature at P is the limiting position of the intersection of the normals at P and P_1 on the curve as P_1 approaches P.*

EXAMPLES

1. Find the radius of curvature of the cycloid

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta).$$

SOLUTION. Since the equation is given parametrically use (6), § 106. Differentiating with respect to θ, we have

$$x' = a(1 - \cos\theta), \qquad y' = a\sin\theta,$$
$$x'' = a\sin\theta, \qquad y'' = a\cos\theta.$$

Then

$$K = \frac{a^2(\cos\theta - \cos^2\theta - \sin^2\theta)}{a^3(2 - 2\cos\theta)^{3/2}} = \frac{-1}{2a\sqrt{2(1-\cos\theta)}} = -\frac{1}{4a\sin(\theta/2)},$$

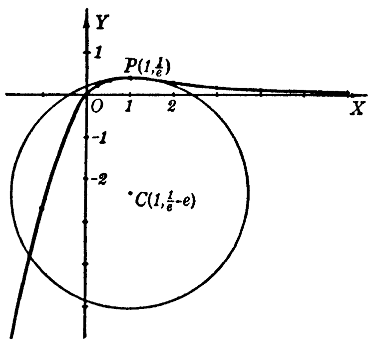


FIG. 116

since $1 - \cos\theta = 2\sin^2(\theta/2)$. Hence

$$R = -4a\sin\frac{\theta}{2}.$$

2. Find the radius of curvature and center of curvature of the curve $y = xe^{-x}$ at its maximum point.

SOLUTION. Differentiating, we have

$$y' = e^{-x}(1 - x), \qquad y'' = e^{-x}(x - 2);$$

so that $y' = 0$ and $y'' = -1/e < 0$ when $x = 1$. Hence $(1, 1/e)$ is the maximum point.

$$R = \frac{[1 + e^{-2x}(1-x)^2]^{3/2}}{e^{-x}(x-2)} \underset{(x=1)}{=} -e.$$

The sign of R shows that the curve is concave downward at the point. We have also

$$h = x - \frac{y'(1+y'^2)}{y''} \underset{(x=1)}{=} 1, \qquad k = y + \frac{1+y'^2}{y''} \underset{(x=1)}{=} \frac{1}{e} - e.$$

As a check, the distance $P(1, 1/e)$ to $C(1, 1/e - e)$ must equal R.

PROBLEMS

Find R for each of the following curves. (Nos. 1–21.)

1. $2xy = a^2$. *Ans.* $(1/a^2)(x^2 + y^2)^{3/2}$.

2. $y = \tan x$ at $x = \pi/4$.

3. $2xy + x + y = a^2$.
Ans. $[(2x+1)^2 + (1+2y)^2]^{3/2}/[4(2x+1)(1+2y)]$.

4. Ellipse, $x^2/a^2 + y^2/b^2 = 1$ at an extremity of the minor axis; at a vertex.

5. Hyperbola, $x^2/a^2 - y^2/b^2 = 1$ at a vertex; at an extremity of the latus rectum. *Ans.* b^2/a; $(b^2/a^4)(a^2 + c^2)^{3/2}$.

6. $y = \log x$ at $(1, 0)$.

7. $y = x \sin (1/x)$ at $x = 2/\pi$. *Ans.* $-16\sqrt{2}/\pi^3$.

8. Catenary, $y = (a/2)(e^{x/a} + e^{-x/a})$.

9. Hypocycloid, $x^{2/3} + y^{2/3} = a^{2/3}$. *Ans.* $3\sqrt[3]{axy}$.

10. Cycloid, $x = a \cos^{-1} [(a - y)/a] - \sqrt{2\,ay - y^2}$. *Ans.* $2\sqrt{2\,ay}$.

11. $x = 2 \cos t, \quad y = 3 \sin t$. *Ans.* $-(4 + 9 \operatorname{ctn}^2 t)^{3/2}/(6 \csc^3 t)$.

12. $x = e^{-t} \cos t, \quad y = e^{-t} \sin t$ at $t = 0$.

13. $x = 4 \cos^3 \theta, \quad y = 4 \sin^3 \theta$ where R is a maximum. *Ans.* 6.

14. $x = t + \cos t, \quad y = 4 - \sin t$ at $t = 0$.

15. Involute, $x = a(\cos \theta + \theta \sin \theta), \quad y = a(\sin \theta - \theta \cos \theta)$. *Ans.* $a\theta$.

16. $r = a \sin 2\,\theta$.

17. $r = a \cos^3 (\theta/3)$. *Ans.* $(3\,a/4) \cos^2(\theta/3)$.

18. $r = a(1 - 2 \cos \theta)$.

19. $r = 4(1 - \sin \theta)$ at $\theta = \pi/6$. *Ans.* 8/3.

20. $r^2 = a^2 \cos 2\,\theta$.

21. $r^2 = a^2 \sec 2\,\theta$. *Ans.* $-r^3/a^2$.

Evaluate the radius of curvature at the point indicated for each curve; draw the curve and the corresponding circle of curvature. (Nos. 22–29.)

22. $x^2 = 4\,y$ at $(2, 1)$.

23. $x = 4 + 2\,y - y^2$ at $(1, -1)$. *Ans.* $17\sqrt{17}/2$.

24. $x^2 + 4\,y^2 = 12$ at $(2, \sqrt{2})$.

25. $x = \cos 2\,t, \quad y = \cos 4\,t$ at $t = \pi$. *Ans.* $17\sqrt{17}/4$.

26. $x = t^3 + 3\,t, \quad y = 3\,t^2$ at $t = 1$.

27. $x = a \sin^3 \theta, \quad y = a \cos^3 \theta$ at $\theta = \pi/4$. *Ans.* $3\,a/2$.

28. At the point (10, 5) on the parabola whose vertex is at the origin and whose axis lies along the x axis.

29. At the point where $y = x$ cuts the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$. Is there an extreme value for R at this point? *Ans.* Yes, $a/\sqrt{2}$.

30. Find the point of the cycloid which has the largest radius of curvature. Is there a point of least R? Explain the test for extreme value.

31. Prove that the normal acceleration of a point moving along the curve $x = g(t)$, $y = f(t)$ is $a_N = v^2/R$, where v is the velocity of the moving point, and R is the radius of curvature of the curve.

32. A point is moving along the arc of the curve $2\,y = x^3$ at the rate of 8 ft./sec. Approximate its change in direction during the half second after it passed the point (2, 4).

33. Approximate by differentials the change in the inclination of the curve $4y = x^2$ by moving along the arc of the curve from the point (2, 1) a distance $\Delta s = 0.1$ unit. *Ans.* $1° 0.9'$.

111. Evolutes. *The locus of the centers of curvature of a given curve is called the* ***evolute*** *of that curve.*

Let the equation of the given curve be

$$(1) \qquad y = f(x).$$

We have found (§ 108) the coordinates of the center of curvature C corresponding to a point $P(x, y)$ on the curve. As P moves along the curve, the point C varies and traces some locus. Call its coordinates $(\bar{x}, \bar{y})$; then

$$(2) \qquad \bar{x} = x - \frac{y'(1+y'^2)}{y''}, \qquad \bar{y} = y + \frac{1+y'^2}{y''},$$

may be considered as *parametric equations of the evolute*, each being expressed in terms of the parameter x. For, since the coordinates of P must satisfy (1), y and its derivatives y' and y'' are all functions of x, and each value assigned to x determines one or more points P by (1), and corresponding points C by (2).

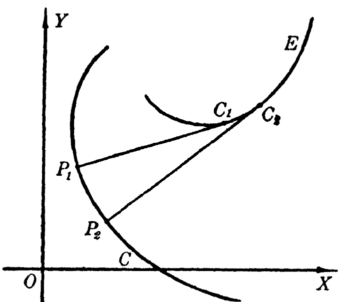


FIG. 117

The rectangular equation of the evolute is found by eliminating x and y between equations (2) and (1). In some problems this is readily done by solving equations (2) for x and y, respectively, in terms of $\bar{x}$ and $\bar{y}$, and then substituting the resulting expressions in (1).

The slope of the evolute at any point is

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\dfrac{d\bar{y}}{dx}}{\dfrac{d\bar{x}}{dx}}.$$

Differentiating equations (2) with respect to x, we find

$$\frac{d\bar{x}}{dx} = 1 - \frac{y''^2 + 3y'^2y''^2 - y'y''' - y'^3y'''}{y''^2}$$

$$= \frac{y'(y'^2y''' + y''' - 3y'y''^2)}{y''^2},$$

and

$$\frac{d\bar{y}}{dx} = y' + \frac{2\,y'y''^2 - y''' - y'^2y'''}{y''^2}$$
$$= \frac{3\,y'y''^2 - y''' - y'^2y'''}{y''^2};$$

hence

$$\frac{d\bar{y}}{d\bar{x}} = -\frac{1}{y'}. \tag{3}$$

But y' is the slope of the tangent to the given curve (1) at the point P. Therefore we have the following theorem.

The tangent to the evolute at any point C is the normal to the given curve at the corresponding point P. That is, the radius of curvature PC at any point is tangent to the evolute at C.

When referred to the evolute, the given curve $y = f(x)$ is called an ***involute.***

EXAMPLE

Find the equation of the evolute of the parabola $y^2 = 2\,px$.

SOLUTION. $y' = \frac{p}{y}, \qquad y'' = -\frac{p}{y^2}\,y' = -\frac{p^2}{y^3}.$

Then

$$\bar{x} = x - \frac{y'(1 + y'^2)}{y''} = x + \frac{y^2}{p} + p = 3\,x + p,$$
$$\bar{y} = y + \frac{1 + y'^2}{y''} = y - \frac{y^3}{p^2} - y = -\frac{y^3}{p^2},$$

or

$$x = \frac{\bar{x} - p}{3}, \qquad y = -(p^2\bar{y})^{1/3}.$$

Substituting these values in $y^2 = 2\,px$, we obtain

$$(p^2\bar{y})^{2/3} = \frac{2\,p}{3}(\bar{x} - p),$$

or

$$27\,p\bar{y}^2 = 8(\bar{x} - p)^3.$$

PROBLEMS

Find the center of curvature for any point of the following curves. (Nos. 1–12.)

1. $y^2 = 4\,x$. *Ans.* $(3\,x + 2,\ -y^3/4)$.

2. $y = xe^{-x}$ at $x = -1$.

3. $2\,xy = a^2$. *Ans.* $[(3\,x^2 + y^2)/2\,x,\ (x^2 + 3\,y^2)/2\,y]$.

4. $y = x \cos x$ at $x = \pi$.

5. $y = \log x$ for minimum R. *Ans.* $[2\sqrt{2},\ -(1/2)(3 + \log 2)]$.

6. $x = t - t^2, \quad y = t + t^2$ at $t = -1$.

7. $x = a \cos \theta, \quad y = b \sin^3 \theta$ at $\theta = \pi/2$. *Ans.* $[0,\ b - a^2/(3\,b)]$.

8. $x = \sin 2\,t, \quad y = \cos^2 t$.

9. $x = e^{-2t} \cos 2\,t, \quad y = e^{-2t} \sin 2\,t$ at $t = 0$. *Ans.* $(0,\ -1)$.

10. $r = 1 - 2 \sin 2\,\theta$.

11. $r^2 = a^2 \sec 2\,\theta$.
Ans. For coincident reference systems $(2\,x^3/a^2,\ -2\,y^3/a^2)$.

12. $r = 1/(1 + \cos \theta),\ \theta = \pi/2$.

Find the equation of the evolute of each of the following curves. (Nos. 13–19.)

13. $y^2 = 8\,x$. *Ans.* $3(-16\,\bar{y})^{2/3} = 8(\bar{x} - 4)$.

14. $x^2 + 4\,y^2 = 4$.

15. $2\,xy = a^2$. *Ans.* $(\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = 2\,a^{2/3}$.

16. $x = 3\,t, \quad y = t^2 + 1$.

17. $x = t - \sin t, \quad y = 1 - \cos t$.
Ans. $\bar{x} = \cos^{-1}(1 + \bar{y}) + \sqrt{-2\,\bar{y} - \bar{y}^2}$.

18. $x = \sin t, \quad y = \cos 2\,t$.

19. $x^{2/3} + y^{2/3} = 4$. *Ans.* $(\bar{x} + \bar{y})^{2/3} + (\bar{x} - \bar{y})^{2/3} = 8$.

ADDITIONAL PROBLEMS

1. Given the curve $y = x/2 + \sin^2 x$.

(*a*) What are the values of x for maximum and minimum values of y?
Ans. $(\frac{7}{12} + n)\pi,\ (\frac{11}{12} + n)\pi$.

(*b*) Find R and K and evaluate for the first positive values of x giving a maximum and a minimum y. *Ans.* $-1/\sqrt{3},\ -\sqrt{3};\ 1/\sqrt{3},\ \sqrt{3}$.

(*c*) What are the values of x for inflections? *Ans.* $\pi/4 + n\pi/2$.

(*d*) Approximate the change in the direction of the curve from $(\pi,\ \pi/2)$ for a change in x of 0.01. *Ans.* $d\alpha = 0.016$ radian.

2. Transform $ds^2 = dx^2 + dy^2$ to polar representation.

3. Find the acute angle for which $3 \tan \theta + \operatorname{ctn} \theta$ is a minimum.
Ans. $\tan^{-1}\sqrt{1/3}$.

4. If $d^2s/dt^2 = k^2s$, is $s = a \sin kt + b \cos kt$ a possible value for s?

5. Where between $\theta = 0$ and $\theta = \pi/2$ does $\sin \theta$ increase half as fast as the arc in the unit circle? *Ans.* $\pi/3$.

6. A barge is being drawn toward a wharf by means of a cable attached to a ring in the floor of the wharf 10 ft. above the windlass on the deck of the barge.

(*a*) Express the length of the cable and the distance of the barge from the wharf as functions of the angle at the windlass between the cable and the horizontal.

(*b*) If the cable is being wound in at the rate of 4 ft./sec., how fast is the angle of (*a*) changing?

(*c*) How fast is the barge approaching the wharf?

(*d*) Evaluate the results of (*b*) and (*c*) when the barge is 24 ft. from the wharf.

(*e*) Does the rate of change of the distance from the barge to the floor of the wharf have a maximum rate of change? If so, where?

(*f*) Does the velocity of the barge have a maximum value? If so, where?

(*g*) Approximate the change in the length of the cable and the distance of the barge from the wharf if the angle changes 0.1° after the discance is 24 ft.

7. Given the curve $y = x - \sin^{-1} x$.

(*a*) Find v along the curve at $x = 1/2$ if $v_y = -2$ units/sec.

Ans. $2\sqrt{(10 - 4\sqrt{3})/(7 - 4\sqrt{3})}$ units/sec.

(*b*) Approximate the change in x for -0.02 unit change in y at $x = 1/2$.

Ans. $dx = (0.02\sqrt{3})/(2 - \sqrt{3})$ units.

(*c*) Approximate the relative and percentage changes in y for a change of x from 0.5 to 0.52. *Ans.* $(2 - \sqrt{3})/(1.18\sqrt{3})$, $(2 - \sqrt{3})/(0.0118\sqrt{3})$.

8. A hoop of radius 2 ft. rolls so that $\omega = 4$ radians per second. An insect at $t = 0$ is at the bottom of the hoop and is crawling along the hoop at the rate of 8 in./sec. Find the path followed by the insect and its velocity along the path.

9. A wheel of radius a rolls along a straight track with constant angular velocity. Find the maximum and minimum velocity of a point on the rim and describe its position for each of these extremes.

Ans. $\theta = 0$, $v = 0$, point at rest on track; $\theta = \pi$, $v = 2\,a\omega$, point moving at top of wheel twice as fast as the hub.

10. A flywheel of radius 5 ft. and fixed center has an angular velocity of 16 radians per second. Find:

(*a*) The parametric equations of a point on the rim.

(*b*) The values of v_x, v_y, and v of a point on the rim.

(*c*) The position of a point if its v_x is a maximum. Find also the value of that maximum.

(*d*) The same as part (*c*) for v_y.

11. A wheel rolls along a straight horizontal road with constant velocity. Show that the highest point of the wheel moves twice as fast as the two points on the rim which are half the radius distant from the road.

12. The equations of motion of a point are

$$x = \log \sin t, \qquad y = 2\sqrt{t}.$$

(*a*) Find v along the path.

(*b*) For what values of t, v, and α does the point cross the y axis?

13. At what point on $16\,x^2 + 9\,y^2 = 400$ does y decrease at the same rate as x increases as a particle moves around the curve clockwise? What is the velocity of the particle at that point? *Ans.* $(3, 16/3)$; $v_x\sqrt{2}$ units/sec.

14. Find the velocity of a point moving along the curve:
(a) $x^2 + 2y^2 = 12$ with $v_y = 2$ units/sec. at $(2, -2)$.
(b) $y = x^3(x-2)^2$ with $v_x = 2$ units/sec. at $(1, 1)$.
(c) $y = \sin x$, with $v_x = 4/\sqrt{5}$ units/sec. at $x = \pi/3$.
(d) $y = 4 - x^2$, with $v_x = -5$ units/sec. at $(1, 3)$. Also find α.

15. A particle moves around a circle of radius a units at b revolutions per second. Find how fast it is moving away from a point c units from the circle. What is the speed when the motion is parallel to the line from the center of the circle to the fixed point?

Ans. $[2\pi ab(a+c)\sin\theta]/\sqrt{2a(a+c)(1-\cos\theta)+c^2}$ units/sec.; $[2\pi ab(a+c)]/\sqrt{2a(a+c)+c^2}$ units/sec.

16. A statue 20 ft. high stands on a pedestal 20 ft. high. If the statue subtends an angle θ at some point in the plane of the base of the pedestal:
(a) At what distance from the pedestal is θ a maximum?
(b) Approximate the change in θ if the distance changes from 10 ft. to 11 ft.

Find K and R for each of the following cases. (Nos. 17–28.)

17. $y^2 = 8x$ at $y = 3$. *Ans.* $R = -7\frac{13}{16}$.

18. $y = e^{-3x}\cos 3x$ at the point $(0, 1)$.

19. $y = x/2 - \sin^2 x$.

$$Ans.\ R = -\frac{(5 - 4\sin 2x + 4\sin^2 2x)^{3/2}}{16\cos 2x}.$$

20. $x = a\cos t,\quad y = b\sin t$ at $t = (1/2)\pi$.

21. $x = \sin t,\quad y = \cos 2t$ at $t = (1/2)\pi$. *Ans.* $R = -17\sqrt{17}/4$.

22. $x = te^{\pi t},\quad y = t^2e^{\pi t}$ at $t = 0$.

23. The cycloid at any point. *Ans.* $R = -4a\sin(\theta/2)$.

24. $x = a\theta - b\sin\theta,\quad y = a - b\cos\theta$. Find the center of curvature at any point.

25. $x = \sin^2\theta,\quad y = \log\cos\theta$ at $\theta = (1/4)\pi$. *Ans.* $R = -\sqrt{2}$.

26. $x = 2(\cos\theta + \theta\sin\theta),\quad y = 2(\sin\theta - \theta\cos\theta)$.

27. $r = e^{a\theta}$. *Ans.* $R = r\sqrt{1+a^2}$.

28. $r^2 = a^2\csc 2\theta$.

29. Find the second and third derivatives of y with respect to x in each of the following cases.
(a) $x = t^2,\quad y = 1 - 2t^3$. *Ans.* $-3/(2t)$, $3/(4t^3)$.
(b) $x = \sin t,\quad y = \cos t$.
(c) $x = t^3,\quad y = e^{3t}$. *Ans.* $e^{3t}(3t-2)/t^5$, $e^{3t}(9t^2 - 18t + 10)/3t^8$.
(d) $x = f(t),\quad y = g(t)$.

30. Derive formula (5) of § 106 directly from formula (4).

31. Express d^3x/dy^3 in terms of derivatives of y.

$$Ans. \ \frac{3\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right)}{\left(\frac{dy}{dx}\right)^5}.$$

32. Light travels from a point A in one medium to a point B in another. A plane surface separates the media. If light has the velocities v_1 and v_2 in the respective media, find the path so the time of passage is a minimum.

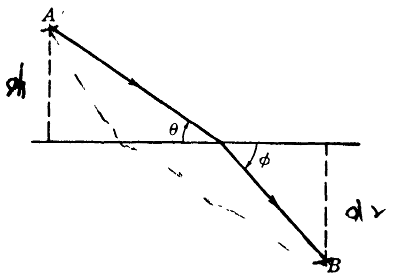


FIG. 118

33. A section of the earth through the poles is an ellipse, the polar radius being 3949.9 mi. and the equatorial radius 3963.3 mi. Using the results for the radius of curvature of an ellipse in Problem 4, § 110, calculate the length of a meridian arc of 1° at the equator, and at the pole.

Ans. 68.708 mi.; 69.405 mi.

CHAPTER VIII

SOLID ANALYTIC GEOMETRY

112. Introduction. The ability to write simple equations for certain surfaces is very important; therefore it seems advisable to insert at this point a short chapter on solid analytic geometry. Its contents may be used as a review of the material which will be used in the following chapters, as it contains sufficient information to permit students to understand the remaining chapters on the calculus.

113. Rectangular Coordinates in Space. A point in space may be located by a set of three real numbers. For ***rectangular coordinates*** these numbers represent the distances of the point from three mutually perpendicular planes which meet in three mutually perpendicular lines called the x axis, y axis, and z axis.

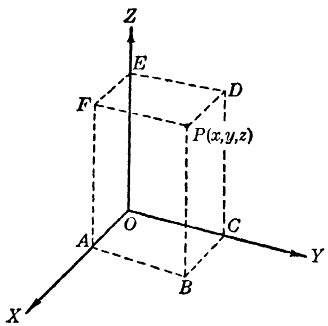


FIG. 119

If x, y, and z represent these distances for the point P of Fig. 119, then $OA = DP = x$, $AB = FP = y$, and $OE = BP = z$. The signs of these coordinates are chosen so as to represent points in all eight octants. Thus, if we consider any point B of the xy plane, it has positive and negative coordinates, as explained for plane cartesian coordinates. Then P has the same x and y coordinates as B; its z coordinate is positive if the point is on one side of the xy plane and negative if on the other. Obviously, this rectangular system of coordinates is a very simple extension of the cartesian coordinates in the plane.

114. Cylindrical Coordinates. Three real numbers locate a point in space just as readily if the points of the plane are located by means of polar coordinates. Thus, we locate a point $B(r, \theta)$ in the horizontal plane and then measure a distance z perpendicular to that plane. This fixes a point $P(r, \theta, z)$, and the system of

coordinates is called ***cylindrical.*** This system is very useful for expressing the locus of points which lie on surfaces of revolution about one of the coordinate axes.

If we take the two sets of axes coincident, so that the positive x axis falls along the polar axis OA and the positive z axis remains the same, we have

(1) $x = r\cos\theta,\quad y = r\sin\theta,\quad z = z,$

and

(2) $r = \sqrt{x^2 + y^2},\quad \theta = \tan^{-1}\dfrac{y}{x},$

and $z = z$ as the relations between the two systems of coordinates. They are used to transform equations from one system of coordinates to the other.

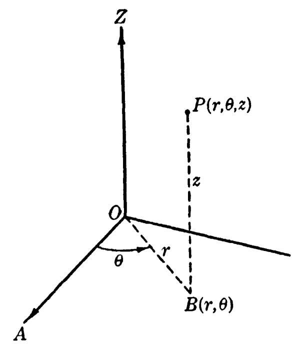


FIG. 120

115. Spherical Coordinates. Let P be any point in space. Then its ***spherical coordinates*** are ρ, θ, and ϕ, where these quantities are defined as shown in the figure. The relations giving the rectangular coordinates in terms of the spherical coordinates are

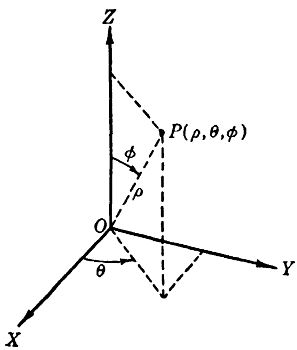


FIG. 121

(3) $$\begin{cases} x = \rho\sin\phi\cos\theta, \\ y = \rho\sin\phi\sin\theta, \\ z = \rho\cos\phi. \end{cases}$$

We observe that the ρ of spherical coordinates is not the r of cylindrical, but that $\rho = r\csc\phi$.

116. Direction Cosines. If a directed line through the origin makes the angles α, β, and γ with the coordinate axes, these angles are called the ***direction angles*** of the line, and their cosines are called the ***direction cosines*** of the line. Any line parallel to such a line and having the same direction has the same direction angles and the same direction cosines.

THEOREM. *If α, β, and γ are the direction angles of a line, then*

(I) $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

PROOF. Let MN represent a line with direction angles α, β, γ.

Draw OP through the origin parallel to MN. Let r represent OP. Evidently

(1) $$x = r \cos \alpha, \quad y = r \cos \beta, \quad z = r \cos \gamma.$$

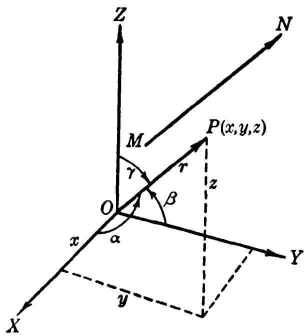


FIG. 122

But

(2) $$x^2 + y^2 + z^2 = r^2.$$

Substituting for x, y, z, we have

$$r^2 \cos^2 \alpha + r^2 \cos^2 \beta + r^2 \cos^2 \gamma = r^2,$$

or, dividing by r^2, we obtain (I).

Any three numbers proportional to the direction cosines of a line are called ***direction numbers*** of the line. If a, b, c are three such numbers,

(3) $$\frac{\cos \alpha}{a} = \frac{\cos \beta}{b} = \frac{\cos \gamma}{c} = k,$$

and substitution of the values of the direction cosines from relations (3) in (I) gives

$$a^2k^2 + b^2k^2 + c^2k^2 = 1.$$

Therefore the proportionality factor is

(4) $$k = \frac{1}{\pm \sqrt{a^2 + b^2 + c^2}}.$$

Hence

(II) $$\begin{cases} \cos \alpha = \dfrac{a}{\pm \sqrt{a^2 + b^2 + c^2}}, \\ \cos \beta = \dfrac{b}{\pm \sqrt{a^2 + b^2 + c^2}}, \\ \cos \gamma = \dfrac{c}{\pm \sqrt{a^2 + b^2 + c^2}}. \end{cases}$$

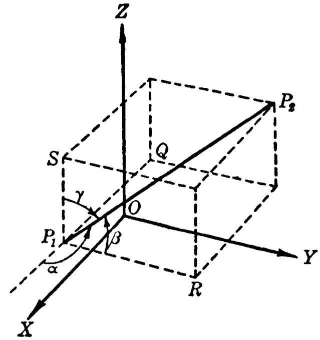


FIG. 123

When a line is directed, the proper choice of sign for k can be made, since its direction angles are fixed.

The numerical values of the direction cosines of the line which passes through the two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are proportional to the projections $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ of the

line-segment P_1P_2 upon the coordinate axes. This is evident from Fig. 123, as

$$\cos \alpha = \frac{P_1Q}{P_1P_2}, \qquad \cos \beta = \frac{P_1R}{P_1P_2}, \qquad \cos \gamma = \frac{P_1S}{P_1P_2},$$

where

$$P_1Q = x_2 - x_1, \qquad P_1R = y_2 - y_1, \qquad P_1S = z_2 - z_1,$$

are the projections of P_1P_2 on the coordinate axes.

THEOREM. *The length of P_1P_2 is given by the formula*

$$\text{(III)} \qquad L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

PROOF. From Fig. 123, we have

$$L^2 = \overline{P_1Q}^2 + \overline{P_1R}^2 + \overline{P_1S}^2,$$

and substitution of the values of the line-segments gives the theorem.

117. The Angle between Two Lines. Suppose two lines through the origin are l_1 and l_2 with direction angles α_1, β_1, γ_1, and α_2, β_2, γ_2, respectively. Take P and Q as any pair of points, one on each line. We define ***the angle*** between two lines as that angle between the positive directions of the lines. Then

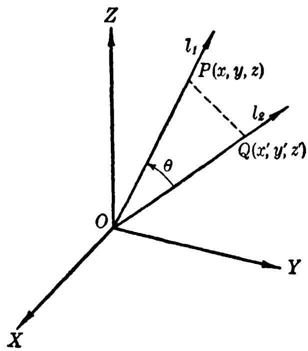


FIG. 124

$$\text{(1)} \qquad \cos \theta = \frac{\overline{OP}^2 + \overline{OQ}^2 - \overline{QP}^2}{2\,\overline{OP}\cdot\overline{OQ}}.$$

Since

$$\overline{OP}^2 = x^2 + y^2 + z^2,$$
$$\overline{OQ}^2 = x'^2 + y'^2 + z'^2,$$

and

$$\overline{QP}^2 = (x - x')^2 + (y - y')^2 + (z - z')^2,$$

$$\cos \theta = \frac{xx' + yy' + zz'}{\sqrt{x^2 + y^2 + z^2}\cdot\sqrt{x'^2 + y'^2 + z'^2}},$$

or

$$\textbf{(IV)} \qquad \cos \theta = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2.$$

If l_1 and l_2 are perpendicular, $\cos\theta = \cos 90° = 0$, whence, for two perpendicular lines,

$$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0. \tag{2}$$

However, x, y, z and x', y', z' are direction numbers of the two lines since they are drawn through the origin. Hence

$$xx' + yy' + zz' = 0. \tag{3}$$

Therefore *the sum of the products of corresponding direction numbers of two perpendicular lines is zero.*

Evidently *the ratios of corresponding direction numbers of two parallel lines are equal.*

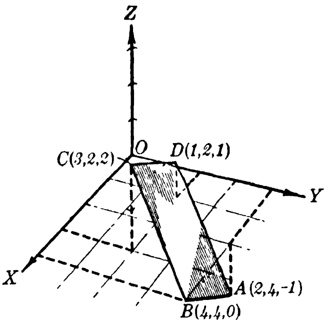


FIG. 125

EXAMPLE

What kind of quadrilateral has the vertices $(2, 4, -1)$, $(4, 4, 0)$, $(3, 2, 2)$, $(1, 2, 1)$ in the order given?

SOLUTION. The direction numbers of the sides of the quadrilateral are: of AB, $2, 0, 1$; of BC, $1, 2, -2$; of CD, $2, 0, 1$; of DA, $1, 2, -2$.

Since AB and CD have direction numbers which are proportional, these lines are parallel. The same is true about BC and DA. Since the sum of the products of corresponding direction numbers of AB and BC is zero, these lines are perpendicular. Hence $ABCD$ is a rectangle.

PROBLEMS

1. Find the direction angles of the line which makes equal angles with the coordinate axes. *Ans.* $54° 44.1'$.

2. Find the direction cosines of a line with $4, -3, 1$ as direction numbers.

3. If $\alpha = 45°$, $\beta = 75°$, what is the value of γ? *Ans.* $48° 51'$, $131° 9'$.

4. What kind of a triangle has the vertices $(7, 3, 4)$, $(1, 0, 6)$, $(4, 5, -2)$?

5. Find the angle between the lines with direction cosines $1/\sqrt{14}$, $2/\sqrt{14}$, $3/\sqrt{14}$ and $2/\sqrt{30}$, $-1/\sqrt{30}$, $5/\sqrt{30}$. *Ans.* $42° 57'$.

6. Find the angle between the lines with $2, 1, -2$, and $1, -1, 1$ as direction numbers.

7. Find the length of the sides of the triangle with vertices $(3, 1, -3)$, $(1, -2, 1)$, and $(1, 0, -3)$. *Ans.* $\sqrt{29}$, $2\sqrt{5}$, $\sqrt{5}$ units.

8. Are the points $(2, -4, 3)$, $(-6, 12, -9)$, $(8, -16, 12)$ on a straight line?

9. Find the coordinates of a point 8 units from the origin if $\alpha = \beta = 60°$ and $\gamma = 45°$. *Ans.* $(4, 4, 4\sqrt{2})$.

10. Find the coordinates of a point 4 units from the origin if $\alpha = 120°$, $\beta = 45°$.

11. Find the angles of the triangle with vertices $(-1, 3, 5)$, $(-2, -1, -1)$, $(2, 3, -4)$. *Ans.* $42° 24'$, $87° 32.5'$, $50° 3.5'$.

12. Find the angles of the triangle with vertices at $(1, -3, 2)$, $(0, 2, -1)$, and $(-1, 4, 1)$.

118. Surfaces. Space Curves. If we write an equation connecting the three variables of either system of coordinates defined previously, points whose coordinates satisfy the equation lie on a ***surface.*** For, let an equation, in rectangular coordinates for instance, be solved for z, so that it takes the form

$$(1) \qquad z = f(x, y).$$

Points of the locus defined by (1) may be located by giving x and y special values and then computing the resulting values for z. If $f(x, y)$ is an algebraic function, the values of z are finite in number and in general are distinct. The fact that they are either distinct or equal shows that the lines through the points on the xy plane perpendicular to that plane meet the locus in points and hence the locus has no thickness. This type of locus is called a ***surface.***

Of course equations may be formed which are not satisfied by the coordinates of any point in space, as was the case in the plane. We are not interested in any such at present.

A ***space curve*** may be regarded as the intersection of any two surfaces which pass through it. The equations $z = f_1(x, y)$ and $z = f_2(x, y)$ of the surfaces considered as simultaneous are equations of the curve of intersection.

119. Cylinders. A surface generated by a straight line which moves parallel to a fixed line and which intersects a given curve is called a ***cylinder.*** The straight line is called the ***generatrix*** and the curve is the ***directrix.*** If the directrix is taken as $f(x, y) = 0$, say, and if the generatrix is perpendicular to the xy plane, the cylinder is represented merely by the equation

$$(1) \qquad f(x, y) = 0$$

When an equation contains two variables only, we plot the locus

as though it represented points in the given plane. Then the locus is formed of all the straight lines through the points of the plane locus parallel to the axis of the missing variable. Such special equations represent ***cylinders.*** If two variables are missing, the locus represents planes parallel to the plane of the missing variables and through the points of the axis corresponding to the roots of the given equation.

120. Surfaces of Revolution. If a plane curve is revolved about a fixed line in its plane, the curve generates a ***surface of revolution*** whose axis is the fixed line. The following examples illustrate a very easy method of deriving the equation of such a surface.

EXAMPLES

1. Find the equation of the surface generated by revolving the ellipse $x^2/a^2 + z^2/b^2 = 1$ about its minor axis.

SOLUTION. The equation of the ellipse means that for any point $P(\bar{x}, \bar{z})$ on the curve we have

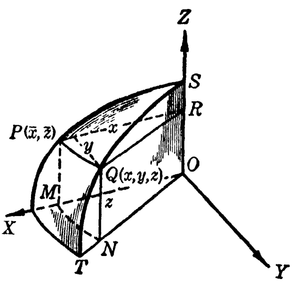


FIG. 126

$$(1) \qquad \frac{\overline{RP}^2}{a^2} + \frac{\overline{MP}^2}{b^2} = 1, \quad \text{or} \quad \frac{\bar{x}^2}{a^2} + \frac{\bar{z}^2}{b^2} = 1.$$

Now suppose the first quadrant of the ellipse revolved about the z axis to some position ST, so that P takes the position $Q(x, y, z)$. Then RP moves to RQ and MP to NQ. Hence equation (1) becomes

$$(2) \qquad \frac{\overline{RQ}^2}{a^2} + \frac{\overline{NQ}^2}{b^2} = 1.$$

But $\overline{RQ}^2 = \bar{x}^2 = x^2 + y^2$ and $\overline{NQ}^2 = \bar{z}^2 = z^2$, where x, y, z are the coordinates of Q, any point of the surface generated by the ellipse.

Substituting in (2), we find the desired equation to be

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

2. What is the equation of the surface generated by revolving the parabola $z^2 = 4x$ about the line $x = 2$ in the xz plane?

SOLUTION. From the equation of the parabola and Fig. 127, we have, for any point $P(\bar{x}, \bar{z})$,

$$(3) \qquad \overline{MP}^2 = 4\,SP = 4(2 - PR).$$

Suppose the parabola revolved about the line $x = 2$ until P moves to the position $Q(x, y, z)$. Then $MP = \bar{z}$ becomes NQ, and PR becomes QR. Draw QT perpendicular to PR. Now $NQ = z$, and, from the right triangle QRT, we have

$$\overline{QR}^2 = \overline{TR}^2 + \overline{TQ}^2 = (2 - x)^2 + y^2,$$

where x, y, z are the coordinates of Q, any point of the surface generated by the parabola.

Substituting these values in (3), we find

$$z^2 = 4[2 - \sqrt{(2 - x)^2 + y^2}],$$

which, when rationalized, becomes

$$(z^2 - 8)^2 = 16[(2 - x)^2 + y^2].$$

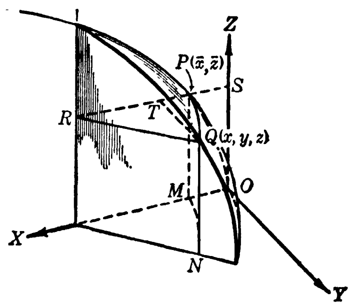


FIG. 127

PROBLEMS

Find the equation of the surface generated by revolving

1. $y^2 = 2\,px$ about its axis. *Ans.* $y^2 + z^2 = 2\,px$.
2. $x^2/a^2 - y^2/b^2 = 1$ about its transverse axis.
3. $x^2/4 + y^2/9 = 1$ about its major axis. *Ans.* $(x^2 + z^2)/4 + y^2/9 = 1$.
4. $x^2 = 8\,y$ about $y = -2$ in the xy plane.
5. $2\,x - 3\,y = 6$ about $x = 5$ in the xy plane. *Ans.* $(3\,y - 4)^2 = 4(x^2 + z^2 - 10\,x + 25)$.
6. $3\,x = 2\,y + 1$ about $x = 1$ in the xy plane.
7. $3\,y = 2\,x^2 + 6$ about $x = 3$ in the xy plane. *Ans.* $216(y - 2) = (2\,x^2 + 2\,z^2 - 12\,x - 3\,y + 6)^2$.
8. $x^2 + y^2 = 4$ about $x = 4$ in the xy plane.
9. $x = y^2 - 2\,y + 3$ about $x = -2$ in the xy plane. *Ans.* $(y^2 - 2\,y + 5)^2 = x^2 + z^2 + 4\,x + 4$.
10. $x^2 = 4\,z$ about $z = 2$ in the xz plane.
11. $x + z = 5$ about $x = 6$ in the xz plane. *Ans.* $(1 + z)^2 = (6 - x)^2 + y^2$.
12. $3\,z^2 = 2\,y + 1$ about $z = 1$ in the yz plane.

121. The Plane. The Line. In the plane ABC, let $P(x, y, z)$ be any point, and let ON be the perpendicular upon the plane from the origin.

Suppose ON is of length p and has the direction angles α, β, γ.

Then the coordinates of N are $p \cos \alpha, p \cos \beta, p \cos \gamma$; hence the direction numbers of NP are

$$x - p \cos \alpha, \qquad y - p \cos \beta, \qquad z - p \cos \gamma.$$

But the lines ON and NP are perpendicular for all positions of P; hence by (3), § 117,

$$\cos \alpha(x - p \cos \alpha) + \cos \beta(y - p \cos \beta) + \cos \gamma(z - p \cos \gamma) = 0.$$

This, when simplified, becomes

(V) $$x \cos \alpha + y \cos \beta + z \cos \gamma = p.$$

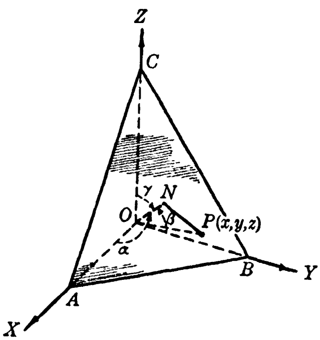


FIG. 128

Since this equation is satisfied by the coordinates of all points of the plane and those points only, we have the following theorem.

THEOREM. *The equation of a plane is always of the first degree in x, y, and z.*

Any equation of the first degree in x, y, and z may be changed into the form of (V) by dividing by the square root of the sum of the squares of the coefficients of the variables. Since p, in (V), is positive, choose the sign for the radical which will make the constant term positive when it is transposed to the right-hand member of the equation.

The equations of any line in space are those of any pair of planes through it. If the line passes through P_1 with direction numbers a, b, and c, its equations may be written in the symmetric form

(VI) $$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

122. The Sphere. *A* ***sphere*** *is a surface each point of which is at a fixed distance R from its center* (h, k, l). Its equation is readily seen to be

(VII) $$(x - h)^2 + (y - k)^2 + (z - l)^2 = R^2.$$

If the center of the sphere is the origin, its equation reduces to

(VIII) $$x^2 + y^2 + z^2 = R^2.$$

If the sphere is tangent to the xy plane at the origin, its center is on the z axis and has the coordinates $(0, 0, \pm R)$. Whence its equation is

$$x^2 + y^2 + (z \pm R)^2 = R^2,$$

or

$$\text{(IX)} \qquad \boldsymbol{x^2 + y^2 + z^2 \pm 2\,Rz = 0,}$$

where the positive sign makes the sphere lie below, and the negative sign above, the xy plane. Similar equations may be written for spheres tangent to other coordinate planes.

Equations (VIII) and (IX) become in cylindrical coordinates

$$\text{(X)} \qquad \boldsymbol{r^2 + z^2 = R^2, \qquad r^2 + z^2 \pm 2\,Rz = 0.}$$

These same equations in spherical coordinates are

$$\text{(XI)} \qquad \boldsymbol{\rho = R, \qquad \rho \pm 2\,R \cos \phi = 0.}$$

123. The Right Circular Cone. *The surface generated by a straight line turning about a fixed point on itself and passing through a given curve is called a* ***cone.***

Suppose the fixed point of the line to be on the z axis and the given curve to be a circle in the $r\theta$ plane with its center at the origin. Then for any point P of the cone, we have, from Fig. 129,

$$\tan \alpha = \frac{QP}{VQ},$$

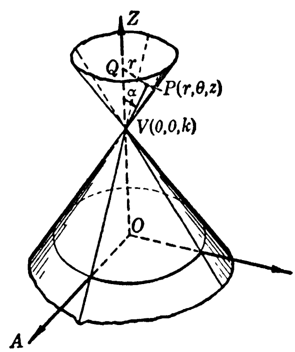


Fig. 129

where α is the angle any position of the generating line makes with the z axis, the ***axis*** of the cone. But $QP = r$ and $VQ = z - k$. Hence the equation of the right circular cone of vertical angle $2\,\alpha$ which has its axis along the z axis and its vertex at $(0, 0, k)$ is

$$\text{(XII)} \qquad \boldsymbol{(z - k) \tan \alpha = r.}$$

When the vertex is at the origin, this equation reduces to

$$\text{(XIII)} \qquad \boldsymbol{r = z \tan \alpha.}$$

These two equations in rectangular coordinates are

$$\text{(XIV)} \qquad x^2 + y^2 = (z - k)^2 \tan^2 \alpha, \qquad x^2 + y^2 = z^2 \tan^2 \alpha,$$

respectively.

In spherical coordinates, the same equations are

$$\text{(XV)} \qquad (\rho \cos \phi - k) \tan \alpha = \rho \sin \phi, \qquad \phi = \alpha.$$

It is worth while to observe that equations (XIV) are composed of terms which are squares of linear expressions in x, y, and z, respectively, and one of the terms is of different sign from the other two when all are written in the same member of the equation. These two characteristics are sufficient for an equation to represent a cone with its axis parallel to one of the coordinate axes.

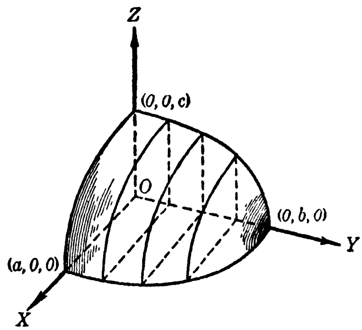


FIG. 130

124. The Ellipsoid. The surface given by

$$\text{(XVI)} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is called an ***ellipsoid.*** It has the following properties:

(*a*) The intercepts on the coordinate axes are $x = \pm a$, $y = \pm b$, $z = \pm c$.

(*b*) The coordinate planes cut the surface in curves called ***traces.*** They are the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1.$$

(*c*) The equation of any section of the surface made by a plane parallel to one of the coordinate planes is that of an ellipse. Thus, the section in the plane $y = k$ is

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 - \frac{k^2}{b^2},$$

which may be written

$$\frac{x^2}{\frac{a^2}{b^2}(b^2 - k^2)} + \frac{z^2}{\frac{c^2}{b^2}(b^2 - k^2)} = 1.$$

As k changes from 0 to b or 0 to $-b$, the axes of the ellipses cut out by the plane $y = k$ decrease and at $k = \pm b$ they become points. These plane sections of a surface, parallel to one of the coordinate planes, afford the best method for sketching a surface. Several such sections are shown in Fig. 130, which represents one-eighth of the ellipsoid.

If any two of the constants a, b, c are equal, the surface is an ***ellipsoid of revolution,*** or a ***spheroid.***

125. The Hyperboloid of One Sheet. The surface represented by

$$\text{(XVII)} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is a ***hyperboloid of one sheet*** and has the following properties:

(a) The intercepts on the x axis are $x = \pm a$, and $y = \pm b$ on the y axis; but the surface does not meet the z axis.

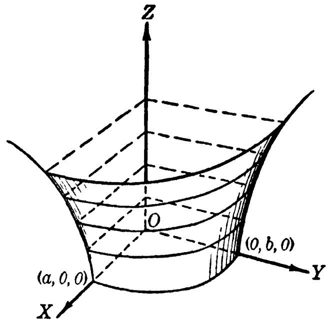


FIG. 131

(b) The traces in the coordinate planes are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \qquad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1.$$

The first of these is an ellipse and the other two are hyperbolas.

(c) The equation of the section in the plane $z = k$ is

$$\frac{x^2}{\frac{a^2}{c^2}(c^2 + k^2)} + \frac{y^2}{\frac{b^2}{c^2}(c^2 + k^2)} = 1,$$

which is an ellipse for all values of k; the axes of the ellipses cut out by such planes increase indefinitely as $k \to \pm \infty$.

In sketching these surfaces it is best to draw the traces in the coordinate planes; then draw plane sections parallel to one of these traces, choosing sections which are closed curves (ellipses or circles) if such exist. Figure 131 shows one-eighth of the surface with several elliptic sections.

If $a = b$, the surface is a ***hyperboloid of revolution of one sheet*** and the closed sections are circles.

This surface has an interesting property which makes it useful for gears. By writing (**XVII**) in the form

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2},$$

both members can be factored into linear factors. These factors permit us to consider the planes

$$(1) \qquad \frac{x}{a} - \frac{z}{c} = k\left(1 - \frac{y}{b}\right), \qquad \frac{x}{a} + \frac{z}{c} = \frac{1}{k}\left(1 + \frac{y}{b}\right).$$

If a point $P_1(x_1, y_1, z_1)$ is on the intersection of these two planes, its coordinates satisfy their equations; hence

$$\frac{x_1}{a} - \frac{z_1}{c} = k\left(1 - \frac{y_1}{b}\right), \qquad \frac{x_1}{a} + \frac{z_1}{c} = \frac{1}{k}\left(1 + \frac{y_1}{b}\right).$$

Multiplying the corresponding members of these two relations and transposing, we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{z_1^2}{c^2} = 1.$$

This means that the point P_1 lies on the surface represented by equation (**XVII**). Therefore, since P_1 is any point of the line of intersection of the two planes, their line of intersection lies on the surface. With k as a parameter, the equations (1) represent a system of lines on the hyperboloid of one sheet. This same conclusion can be reached by means of the system of lines

$$(2) \qquad \frac{x}{a} - \frac{z}{c} = k\left(1 + \frac{y}{b}\right), \qquad \frac{x}{a} + \frac{z}{c} = \frac{1}{k}\left(1 - \frac{y}{b}\right);$$

hence the hyperboloid of one sheet may be regarded as a ***ruled surface*** with two sets of rulings. No two lines of the same system meet, but each line of either system meets all of the other.

126. The Hyperboloid of Two Sheets. The surface represented by the equation

$$(\mathbf{XVIII}) \qquad -\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is called a ***hyperboloid of two sheets.*** Evidently the intercepts on the y axis are $y = \pm b$ and the surface does not meet the x or z axes.

The traces in the xy and yz planes are hyperbolas with equations

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1, \qquad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

respectively. There is no trace in the xz plane.

The equation of a section in a plane parallel to the xz plane, say $y = k$, is

$$\frac{x^2}{\frac{a^2}{b^2}(k^2 - b^2)} + \frac{z^2}{\frac{c^2}{b^2}(k^2 - b^2)} = 1,$$

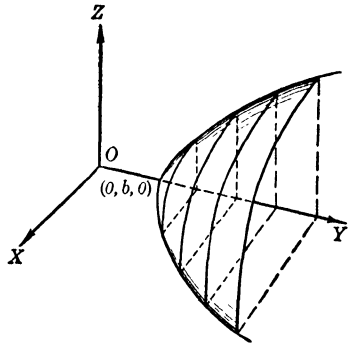


FIG. 132

which is an ellipse if $k > b$ or if $k < -b$. The axes of these sections increase indefinitely as $k \rightarrow \pm \infty$. For $k = \pm b$, the section has the equation

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 0,$$

which is a ***point.*** There is no section corresponding to $y = k$ for $-b < k < b$.

One-eighth of the surface is shown in Fig. 132, drawn by means of sections made by planes $y = k_i$.

If $a = c$, the surface is a ***hyperboloid of revolution of two sheets.***

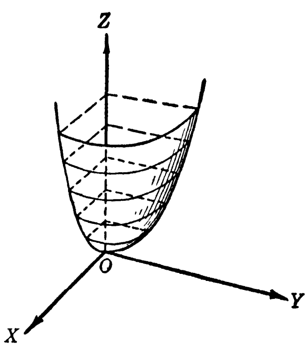


FIG. 133

127. The Elliptic Paraboloid. If taken as shown in Fig. 133, this surface has the equation

$$\text{(XIX)} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = cz.$$

The intercepts are all zero. The traces in the coordinate planes are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0, \qquad \frac{x^2}{a^2} = cz, \qquad \frac{y^2}{b^2} = cz.$$

The first of these is a point and the other two are parabolas. Sections parallel to the xy plane are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = ck,$$

which are ellipses for all values of k such that $ck > 0$. The axes of these sections increase indefinitely as $ck \to +\infty$.

Figure 133 shows one-fourth of the surface for the case of $c > 0$, drawn by means of the closed parallel sections lying in the planes $z = k_i$.

If $a = b$, the surface is a ***paraboloid of revolution.***

128. The Hyperbolic Paraboloid. The surface whose equation is

$$\text{(XX)} \qquad \frac{y^2}{b^2} - \frac{x^2}{a^2} = cz$$

is called a ***hyperbolic paraboloid.*** The surface has all of its intercepts zero.

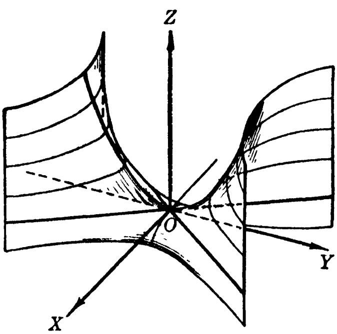


FIG. 134

Its traces in the coordinate planes are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \qquad \frac{x^2}{a^2} = -cz,$$

$$\frac{y^2}{b^2} = cz$$

The first of these may be written

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 0,$$

and therefore represents two straight lines intersecting at the origin. The other two traces are parabolas. All sections in the planes $z = k_i$ are hyperbolas and their axes increase indefinitely as $k_i \to \pm\infty$.

If c and k_i have the same sign, the transverse axes of these hyperbolic sections are parallel to the y axis, and when c and k_i have opposite signs the transverse axes are parallel to the x axis. Figure 134 shows this surface for $c > 0$. Since it does not have closed sections, we should sketch it, as shown, by means of hyperbolas.

129. The Elliptic Cone. If the vertex is placed at the origin and its axis along the z axis, the elliptic cone has the equation

$$\text{(XXI)} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

All intercepts are zero and the traces in the coordinate planes are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0, \qquad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 0, \qquad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

The first of these is a point, and each of the others is a pair of straight lines meeting at the origin.

Sections in the planes $z = k_i$, that is, perpendicular to the axis of the cone, are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k_i^2}{c^2}.$$

These are ellipses which increase in dimensions as $k_i \to \pm \infty$. Figure 135 shows one-fourth of the surface, drawn by means of sections parallel to the xy plane, and the traces in the other planes.

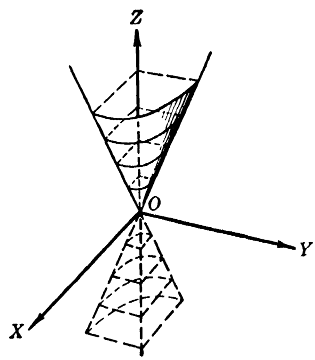


FIG. 135

If $a = b$, the cone is ***circular.***

The equations of the surfaces discussed in this chapter have been given their simplest forms by the proper choice of the coordinate planes.

130. Translation of the Axes. If the axes are translated to the new origin $O'(h, k, l)$, as referred to the x, y, z axes, we have for any point P,

$$\text{(XXII)} \qquad \begin{cases} x = x' + h, \\ y = y' + k, \\ z = z' + l. \end{cases}$$

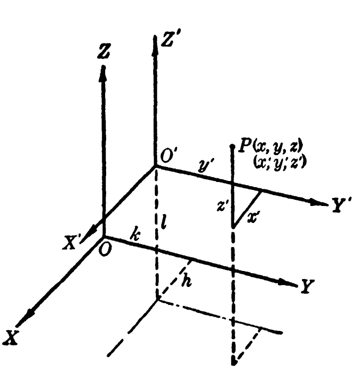


FIG. 136

Suppose the equation of a surface is given in terms of x', y', and z' and it is desired to find the equation of the surface as referred to the x, y, z axes. It is merely necessary to replace the x', y', and z' by their values as given by (XXII).

Thus, in the equations of the preceding articles, we must replace x, y, z by $x - h$, $y - k$, $z - l$ if the point we choose for the origin is at the point (h, k, l). However, the content of these articles

still holds if we consider the planes $x = h$, $y = k$, $z = l$ as reference planes when we find the intercepts, traces, and sections of any surface.

EXAMPLE

What surface has the equation

$$x^2 + 3\,y^2 - 4\,z^2 + 2\,x - 12\,y + 8\,z - 3 = 0?$$

Sketch the surface.

SOLUTION. Since first-degree terms are present in the equation of the surface, it is necessary to complete the squares of each pair of terms involving one variable. This gives

$$(x + 1)^2 + 3(y - 2)^2 - 4(z - 1)^2 = 12.$$

This equation can be written in the form

$$\frac{(x + 1)^2}{12} + \frac{(y - 2)^2}{4} - \frac{(z - 1)^2}{3} = 1.$$

In this form we recognize the equation as that of a hyperboloid of one sheet, as the equation is similar to equation (**XVII**), § 125.

The fact that $x + 1$, $y - 2$, and $z - 1$ replace the x, y, and z of (**XVII**) merely means that the center of the hyperboloid is at the point $(-1, 2, 1)$.

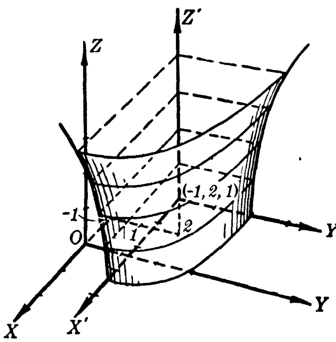


FIG. 137

Draw lines through this point which are parallel to the x, y, and z axes respectively. Then sketch as shown in Fig. 137. Here we have used the traces of the surface in the planes $x = -1$, $y = 2$, $z = 1$ and sections in the planes $z = k_1$. One-eighth of the surface is shown.

PROBLEMS

1. Derive the equation of the plane whose intercepts on the axes are respectively a, b, and c units.

Discuss, sketch, and name the surfaces represented by the following equations. (Nos. 2–33.)

2. $x + y = 4.$

3. $3\,x - 3\,y - 2\,z + 6 = 0.$

4. $x^2 + y^2 + z^2 = 9.$

5. $x^2 + y^2 + z^2 - 4\,z = 0.$

6. $r^2 + z^2 = 16.$

7. $y^2 = 4 - z.$

8. $y^2 = x - 2.$

9. $x^2 + z^2 - 2\,az = 0.$

10. $xy + 3\,x - 4\,y = 3.$

11. $r = 2\cos\theta.$

12. $r = 3\sin\theta.$

13. $r = a\cos 3\,\theta.$

14. $z + r = 0.$

15. $y^2 + z^2 = 2\,x.$

16. $4\,x^2 - y^2 + z^2 = 0.$

17. $r^2 = 1 + z.$

18. $4x - y^2 - 4z^2 = 4$.

19. $z = 6 - 2x^2 - y^2$.

20. $(x - 1)^2 + 4y^2 = 4z^2$.

21. $x^2 + y^2 - z^2 = 16$.

22. $r\sqrt{3} = z$.

23. $x^2 + 4y^2 + z^2 = 4$.

24. $x^2 - 4y^2 + z^2 = 4$.

25. $x^2 - 4y^2 - z^2 = 4$.

26. $6x^2 - 3(y - 4)^2 + 4z^2 = 0$.

27. $r^2 = 4 - z$.

28. $4x^2 + 3y^2 + 4z^2 = 36$.

29. $4x^2 + 3y^2 - 4z^2 = 0$.

30. $x^2/9 - y^2/25 - z^2/25 = -1$.

31. $r = a(1 + \cos\theta)$.

32. $r^2 = a^2 \cos 2\theta$.

33. $r = z^2$.

Draw each of the following lines and find its direction numbers. (Nos. 34–35.)

34. (*a*) $2x - 3y = 0, \quad 5x + 16y = 0$.
(*b*) $x + y + z = 3, \quad x + y = 6$.

35. (*a*) $x + 3y - 3z = 4, \quad 3x + 4y + 6z = 6$.
(*b*) $(x - 4)/6 = (y + 3)/3 = -(z + 2)/2$. *Ans.* (*b*) 6, 3, -2.

36. Find the equation of the plane through (1, 2, 3) and perpendicular to the line $3x + 2y + 1 = 0$, $5x - 2z - 3 = 0$.

Sketch each of the following pairs of surfaces on one reference scheme and show the curve of intersection. (Nos. 37–43.)

37. $x^2 + y^2 = 4z, \quad x^2 + y^2 + z^2 = 9$.

38. $r = 4\cos\theta, \quad r = z$.

39. $r = 2\sin\theta, \quad r^2 + z^2 = 4$.

40. $3x^2 + 4y^2 = -2z, \quad x^2 + y^2 - 2(z + 2) = 0$.

41. $y^2 = 4z, \quad x^2 = 4z$.

42. $x^2 + z^2 = 9, \quad y^2 + z^2 = 9$.

43. $r^2 = 3z, \quad r\cos\theta + z = 2$.

44. Write the equation of the right circular cone with a vertical angle of 120° at the origin if its axis is along the z axis.

45. Write the relation connecting the three sides of a right triangle. Sketch the relation. *Ans.* $z^2 = x^2 + y^2$.

46. Write the equation of the right circular cone of vertical angle $\pi/2$ at (0, 0, 2) and axis along the z axis.

47. Write the equation of the right circular cylinder which has the x axis as one element and has (0, 0, 1) as the center of its yz trace.
Ans. $y^2 + (z - 1)^2 = 1$.

48. Write the equation of an ellipsoid whose center is the origin if its semi-axes are 2, 3, 4 units, respectively, and lie along the reference lines.

49. Write the equation of the paraboloid of revolution whose vertex is (0, 0, 2) and which cuts the xy plane in a circle of radius 2 units.

Ans. $4 - 2z = x^2 + y^2$.

50. A paraboloid of revolution is formed by revolving about its axis a parabolic segment whose base and altitude are each 10 units. Choose a reference scheme and derive the equation of the surface.

Find the direction numbers of each of the following lines. (Nos. 51–52.)

51. (*a*) $2x + y - 3z = 7, \quad x - 4y + 3z + 9 = 0.$
(*b*) $3x - 2y + z = 12, \quad x - 5y - 2z = 10.$

Ans. (*a*) 1, 1, 1; (*b*) 9, 7, − 13.

52. (*a*) $2x - y + z = 7, \quad x + y + 2z = 11.$
(*b*) $x + 5y - 3z = 10, \quad 2x - 3y - z = 10.$

Find the equation of the plane determined by each of the following sets of conditions. (Nos. 53–54.)

53. (*a*) Passing through the points (8, − 2, 6), (3, 4, − 3), (2, 2, − 2).
(*b*) Passing through the point (1, 1, 2) and the line $x - 3y - 2z = 0$, $x + y + z - 2 = 0$.

Ans. (*a*) $6x - 7y - 8z - 14 = 0$; (*b*) $4x + z - 6 = 0$.

54. (*a*) Passing through the points (4, 2, 1), (−1, − 2, 2), (0, 4, − 5).
(*b*) Passing through the point (0, 2, −3) and the line $3x - z + 12 = 0$, $x + 3y + 17 = 0$.

Find the equation of the plane through each of the following pairs of intersecting lines. (Nos. 55–56.)

55. $\dfrac{x+2}{3} = \dfrac{3-y}{5} = \dfrac{z-1}{2}$ and $\dfrac{x+2}{-2} = \dfrac{y-3}{4} = \dfrac{z-1}{3}$.

Ans. $23x + 13y - 2z + 9 = 0$.

56. $[3x - 2y = 0, x + 3y - 2z = 3]$ and $[x + y - z = 0, 5y - 3z = 0]$.

CHAPTER IX

PARTIAL DIFFERENTIATION — DIRECTIONAL DERIVATIVES — ENVELOPES

131. Partial Derivatives. A function of two or more independent variables can be differentiated with respect to any one of the variables if the others are considered as constants during the operation. Such a derivative is called a ***partial derivative*** of the function.

If we consider $z = f(x, y)$, the first partial derivative of z with respect to x is defined as follows:

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}; \tag{1}$$

and we represent the same by one of the symbols

$$\frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x} = f_x(x, y).$$

The student should recognize (1) as the definition of the derivative of a function of one variable x; that is what it really means, since y is assumed to remain constant.

Similarly we have

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,y + \Delta y) - f(x, y)}{\Delta y}. \tag{2}$$

If we have a function of more than two independent variables the first partial derivatives of the function with respect to each variable are defined and found just as in the case of $f(x, y)$. All variables are treated as constants except the one with respect to which the derivative is taken.

We are familiar with functions that depend upon two or more variables; for example: the volume or surface of a cone depends upon the radius of its base and its altitude; the volume of a gas depends upon its temperature and the pressure to which it is subjected; the motion of a body is dependent on all the forces acting upon it. Also, in filling a cylindrical can with water we have the volume of the water changing in depth only; in rolling

paper into rolls for printing, the volume changes due to a change in the radius of the roll alone. These are illustrations of functions of more than one dimension, however, only one dimension is allowed to change. Hence the rate of change of the volume in each case is a partial derivative.

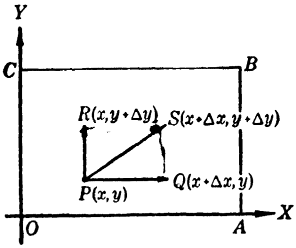


FIG. 138

To illustrate this, let the rectangle $OABC$ of Fig. 138 be a plate heated in such a manner that the temperature at any point $P(x, y)$ is given by some function of x and y. If $T_P = f(x, y)$ then $T_Q = f(x + \Delta x, y)$ as P and Q have the same ordinate. Hence the temperature changes in going from P to Q by the amount

$$T_Q - T_P = f(x + \Delta x, y) - f(x, y).$$

As this change is due to a change in x alone, the temperature along the line PQ has as its rate of change at P the following limit:

$$\lim_{\Delta x \to 0} \frac{T_Q - T_P}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$

which is exactly an illustration of a first partial derivative of a function of two variables and is accordingly $\partial T/\partial x$.

Similarly, the rate of change of T in the direction PR is $\partial T/\partial y$. However, the change of T in any other direction PS will depend upon both Δx and Δy and hence the rate of change is not a partial derivative.

EXAMPLES

1. If $T = e^x \sin y$, find $\partial T/\partial x$, $\partial T/\partial y$ at the point $(1, 2)$.

SOLUTION. $T_x = e^x \sin y$ and at $(1, 2)$ is $e \cdot \sin 2$ or 2.4717. Also $T_y = e^x \cos y$ and at $(1, 2)$ this is -1.1312.

2. Find the rate of change of the volume of a right circular cone with respect to the radius of its base when the radius is 4 inches, if its altitude remains 6 inches.

SOLUTION. The volume of the cone is $V = (1/3)\pi r^2 h$ and we want $\partial V/\partial r$ evaluated for $r = 4''$ and $h = 6''$. Since

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h,$$

the desired quantity is $(2/3)\pi \cdot 4 \cdot 6 = 16\pi$ cu. in. per in.

132. Higher Partial Derivatives. If $z = f(x, y)$ is differentiated twice with respect to x or y, or once with respect to each x and y, we have ***second order partial derivatives.*** These second partial derivatives are represented by the symbols

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y), \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y),$$
$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y), \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y).$$

If $f(x, y)$ is a continuous function with continuous derivatives,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.^*$$

A similar notation is used for higher derivatives than the second of functions of two or more independent variables. Thus

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial x^3} = f_{xxx}(x, y), \quad \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y \partial x}\right) = \frac{\partial^3 z}{\partial y^2 \partial x} = f_{xyy}(x, y),$$
$$\frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial y \partial x^2} = f_{xxy}(x, y), \quad \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial x \partial y^2} = f_{yyx}(x, y),$$
$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x \partial y}\right) = \frac{\partial^3 z}{\partial x^2 \partial y} = f_{yxx}(x, y), \quad \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial y^3} = f_{yyy}(x, y),$$

and similarly for the fourth and higher derivatives.

EXAMPLE

If $z = x^2 y + 2\,xe^{1/y}$, show that $z_{xy} = z_{yx}$.

SOLUTION. The first partial derivatives of z are $z_x = 2\,xy + 2\,e^{1/y}$, and $z_y = x^2 - 2\,xe^{1/y}/y^2$. Differentiating the first of these with respect to y and the second with respect to x, we have

$$z_{xy} = 2\,x - \frac{2\,e^{1/y}}{y^2} = z_{yx}.$$

133. Geometric Representation of z_x and z_y. Tangent Plane. Since $z = f(x, y)$ may be considered to represent a surface, a very simple geometric representation of the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ can be given.

Suppose a portion of the surface $z = f(x, y)$ is represented by $ABCD$ of Fig. 139.

The plane $x = x_1$ cuts the surface in the curve QN and LM is

* For a proof of this see Goursat-Hedrick, *Mathematical Analysis*, Vol. I, § 11.

cut out by the plane $y = y_1$. Then WP_1 is the value of z at the intersection of the curves QN and LM. If we let $WE = P_1K = \Delta x$ and $WH = P_1F = \Delta y$, then $WP_1 = f(x_1, y_1)$, $EM = f(x_1 + \Delta x, y_1)$, and $HN = f(x_1, y_1 + \Delta y)$. Whence

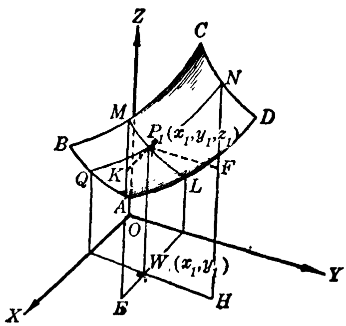


FIG. 139

$$KM = f(x_1 + \Delta x, y_1) - f(x_1, y_1),$$

and

$$FN = f(x_1, y_1 + \Delta y) - f(x_1, y_1).$$

This means that

$$\frac{\partial z}{\partial x} = \lim_{P_1K \to 0} \frac{KM}{P_1K}$$

is the slope of the curve LM at the point P_1. And similarly

$$\frac{\partial z}{\partial y} = \lim_{P_1F \to 0} \frac{FN}{P_1F}$$

is the slope of QN at P_1.

Hence *the first partial derivatives of $z = f(x, y)$ are the slopes of the curves cut from the surface by sets of planes which are parallel to the yz and xz planes, respectively.*

It is now easy to find the equation of the tangent plane to the surface at P_1. This plane intersects the planes $x = x_1$ and $y = y_1$ in lines tangent to QN and ML at P_1.

These tangent lines are

(1) $$z - z_1 = \frac{\partial z_1}{\partial y_1}(y - y_1), \qquad x - x_1 = 0,$$

and

(2) $$z - z_1 = \frac{\partial z_1}{\partial x_1}(x - x_1), \qquad y - y_1 = 0.$$

The equation of any plane through P_1 is

(3) $$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.$$

If this plane contains the tangent line (1), whose direction numbers are 0, 1, $\partial z_1/\partial y_1$, we must have

$$B + C\frac{\partial z_1}{\partial y_1} = 0.$$

Similarly, if (3) contains the line (2), we must have

$$A + C\frac{\partial z_1}{\partial x_1} = 0.$$

Hence*

$$\frac{\partial z_1}{\partial x_1} = -\frac{A}{C}, \qquad \text{and} \qquad \frac{\partial z_1}{\partial y_1} = -\frac{B}{C}.$$

Solving (3) for $z - z_1$ and substituting these values, *the equation of the tangent plane* becomes

$$(4) \qquad z - z_1 = \frac{\partial z_1}{\partial x_1}(x - x_1) + \frac{\partial z_1}{\partial y_1}(y - y_1).$$

EXAMPLES

1. If $z = 3x^2 - 2y^2 + 2xy$, find $\partial z/\partial x$ and $\partial z/\partial y$.

SOLUTION. Differentiating z as though x were the only variable in it, we find $\partial z/\partial x = 6x + 2y$. Similarly for y, $\partial z/\partial y = 2x - 4y$.

2. Find the equation of the tangent plane to the paraboloid $z = 2x^2 + 4y^2$ at the point $(1, 1, 6)$.

SOLUTION. At the point $\partial z/\partial x = 4$ and $\partial z/\partial y = 8$. Hence the tangent plane is $z - 6 = 4(x - 1) + 8(y - 1)$ or

$$4x + 8y - z - 6 = 0.$$

3. If x and y are given parametrically by the two independent variables r and θ such that $x = e^{2r}\cos\theta$, $y = e^r \sin\theta$, find x_r, x_θ, y_r, y_θ.

SOLUTION. Evidently $x_r = 2e^{2r}\cos\theta$, $y_r = e^r\sin\theta$. Also we have

$$x_\theta = -e^{2r}\sin\theta,\ y_\theta = e^r\cos\theta.$$

4. Suppose in Example 3 above that x and y are the independent variables. Find r_x, r_y, θ_x, θ_y.

SOLUTIONS. It is essential that the student realize that r and θ are now the dependent variables and each is a function of x and y. Therefore, r and θ have partial derivatives with respect to each x and y. Hence we may solve the given equations for r and θ in terms of x and y and find these partial derivatives just as in the preceding examples. However, such a solution is often inconvenient or more difficult than the method given below. That is, find the partial derivatives of both sides of each of the two equations

$$x = e^{2r}\cos\theta, \qquad y = e^r\sin\theta$$

with respect to x, as is done in implicit differentiation. Recalling that y is assumed constant, we have

$$1 = 2e^{2r}\cos\theta \cdot \frac{\partial r}{\partial x} - e^{2r}\sin\theta \cdot \frac{\partial \theta}{\partial x},$$

$$0 = e^r\sin\theta \cdot \frac{\partial r}{\partial x} + e^r\cos\theta \cdot \frac{\partial \theta}{\partial x}.$$

* These results can be obtained also as follows. The section of (3) by the plane $x = x_1$ is obtained by setting $x - x_1 = 0$ in (3). Solve the resulting equation for $z - z_1$ and compare with (1). Proceed similarly for (2).

Since such equations are always linear in the partial derivatives, they are very readily solved. Thus we get

$$\frac{\partial r}{\partial x} = \frac{\cos\theta}{e^{2r}(1+\cos^2\theta)}, \qquad \frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{e^{2r}(1+\cos^2\theta)}.$$

These results may be expressed in terms of x and y with the help of the original equations. They become

$$\frac{\partial r}{\partial x} = \frac{2x}{(y^2+\sqrt{y^4+4x^2})\sqrt{y^4+4x^2}},$$

$$\frac{\partial\theta}{\partial x} = -\frac{y\sqrt{2}}{\sqrt{y^2+\sqrt{y^4+4x^2}}\sqrt{y^4+4x^2}}.$$

Similarly, $\partial r/\partial y$ and $\partial\theta/\partial y$ may be found.

PROBLEMS

Find the partial derivatives specified below. (Nos. 1–5.)

1. $z = 6 - 2x^2 - y^2$, find z_x, z_y, z_{xy}. *Ans.* $-4x$, $-2y$, 0.

2. $u = x^2 + 5xy^2$, find $xu_{xx} + (1/2)yu_{yx}$.

3. $4x^2 - y^2 + z^2 = 0$, find z_x, z_y, z_{xx}, z_{yx} at $(2, 5, 3)$.
Ans. $-8/3$, $5/3$, $-100/27$, $40/27$.

4. $u = x^2 - 2xy + 3y^2$, find u_{yx}, u_{xy}.

5. $x^3yz^4 - 2y^2z^3 + x^2y^2 + 2 = 0$, find z_x, z_y.
Ans. $(2xy + 3x^2z^4)/(6yz^2 - 4x^3z^3)$,
$(2x^2y + x^3z^4 - 4yz^3)/(6y^2z^2 - 4x^3yz^3)$.

6. $u = (x^2 + y^2)/(y^2 - x^2)$, show that $u_{xy} = u_{yx}$.

7. $u = x^3y^2 - 2xy^4 + 3x^2y^3$, show that $xu_x + yu_y = 5u$.

8. $u = (x^2 + y^2)^{1/3}$, show that $3xu_{yx} + 3yu_{yy} + u_y = 0$.

9. $u = (ax + by + cz)^n$, show that $xu_x + yu_y + zu_z = nu$.

10. $z = \log\sqrt{x^2 + y^2}$, find $z_{xx} + z_{yy}$.

11. $z = x^2 \sin 2y$, show that $z_{xy} = z_{yx}$.

12. $u = \sin x^2y$, is $u_{xy} = u_{yx}$?

13. $x = r\cos\theta$, $y = r\sin\theta$, find x_r, y_θ. *Ans.* $\cos\theta$, $r\cos\theta$.

14. $x = e^r\sin\theta$, $y = e^r\cos\theta$, find x_r, y_θ.

15. $x = r\cos\theta$, $y = r\sin\theta$, find r_x, θ_y. *Ans.* x/r or $\cos\theta$; x/r^2 or $(\cos\theta)/r$.

16. $x = e^r\sin\theta$, $y = e^r\cos\theta$, find r_y, θ_x.

17. $z = e^{-y^2}\cos x$, show that $z_{yy} - 2z_{xx} = 4y^2z$.

18. $z = e^{2x}(\cos^2 y - \sin^2 y)$, find $z_{xx} + z_{yy}$.

19. $z = (x^2 + y^2) \tan^{-1}(x/y)$, find z_x, z_y, z_{yx}.

Ans. $y + 2x \tan^{-1}(x/y)$, $2y \tan^{-1}(x/y) - x$, $(y^2 - x^2)/(x^2 + y^2)$.

20. $z = e^{y/x} \sin (y/x)$, find $yz_y + xz_x$.

21. $z = x \sin y - ye^x$, find z_{yx}. *Ans.* $\cos y - e^x$.

22. $z = x^2 e^{-2y} + y \log [3 - \csc (x/2)] + \sin y \tan^{-1} x$, find z_x, z_y, z_{yy}.

23. Given the surface $z = 2x^2 + 3y^2 - 4$, find the slopes of the curves cut from this surface by planes through $(1, -1, 1)$ parallel to the xz and yz planes.

Ans. $4, -6$.

24. Given the surface $z = x^2 - y^2 + 3x - 7$, find the slopes of the curves cut from this surface by the planes $x = -3$, $y = -2$.

25. Use the area of a triangle in terms of two sides and the included angle to find the rate of change of the area with respect to the angle; with respect to either side. *Ans.* $(1/2)xy \cos \theta$, $(1/2)y \sin \theta$, $(1/2)x \sin \theta$.

26. Using the law of cosines, derive the rate of change of one side with respect to the opposite angle.

27. If the resistance of the air is not neglected in considering vibrations of strings, the equation $y_{tt} + 2ky_t = a^2 y_{xx}$ occurs. Is the following function, $y = e^{-kt} \sin \alpha x \cos (t\sqrt{a^2\alpha^2 - k^2})$, a solution of this equation?

28. The equation $z_{tt} = c^2(z_{xx} + z_{yy})$ occurs in the theory of the vibration of stretched membranes. Show that $z = \sin \alpha x \sin \beta y \sin (ct\sqrt{\alpha^2 + \beta^2})$ is a function satisfying this equation.

29. A function defining the potential of a point of a thin sheet due to current flow must satisfy the relation $v_{xx} + v_{yy} = 0$. Are the functions $v = 1 - [\tan^{-1}(y/x)]/\pi$ and $v = -[\log (x^2 + y^2)]/(2\pi)$ solutions of this equation?

Find the equation of the tangent plane to each of the following surfaces at the point indicated. (Nos. 30–39.)

30. $xz + 2x + 4z = 5$, $(2, 5, 1/6)$.

31. $x^2 - y^2 + z^2 = 6$, $(1, 2, -3)$. *Ans.* $x - 2y - 3z = 6$.

32. $x^2 + y^2/4 - z^2/9 = 1$, $(-1, 2, 3)$.

33. $2x + y^2/2 + z^2/4 = 0$, $(-3/8, 1, 1)$. *Ans.* $8x + 4y + 2z = 3$.

34. $x^2 - 4x + 2y = 0$, $(4, 0, 5)$.

35. $xy + yz + zx = 0$, $(2, -3, -6)$. *Ans.* $9x + 4y + z = 0$.

36. $x^2 + y^2 + z^2 - 6z = 0$, $(2, 2, 4)$.

37. $x^2 + y^2 - 3z = 2$, $(-2, -4, 6)$. *Ans.* $4x + 8y + 3z + 22 = 0$.

38. $2x^2 - y^2 - 3z^2 - 4y + 6z + 2 = 0$, $(3, -2, 4)$.

39. $x^2 + 3y^2 - 4z^2 + 2x - 12y + 8z - 7 = 0$, $(1, -2, 4)$.

Ans. $x - 6y - 6z + 11 = 0$.

134. The Increment and Differential of $z = f(x, y)$. If x and y are given increments Δx and Δy, the resulting increment of z is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y). \tag{1}$$

Since $WP = z$ and $JC = z + \Delta z$, we have $\Delta z = JC - WP = RC$.

Now, assuming that z and its derivatives are continuous functions, the point C will approach P as a limit as PK and PF approach zero as a limit in any manner whatsoever.

The line-segment RC is made up of $RT = KM$ and TC. We have shown in the preceding article that

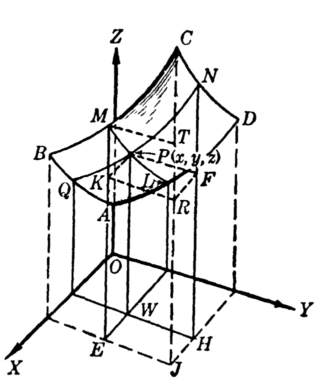


FIG. 140

$$\lim_{PK\to 0} \frac{KM}{PK} = \frac{\partial z}{\partial x}; \tag{2}$$

whence (§ 85)

$$\frac{KM}{PK} = \frac{\partial z}{\partial x} + e_1, \tag{3}$$

where $e_1 \to 0$ as $PK = \Delta x \to 0$. From (3), we have

$$KM = \left(\frac{\partial z}{\partial x} + e_1\right) \Delta x. \tag{4}$$

Also we see that TC is such that

$$\lim_{MT\to 0} \frac{TC}{MT} = \text{slope of } MC.$$

However, as $\Delta x \to 0$, the curve MC approaches coincidence with PN, and TC with FN. Hence if we take the limit of TC/MT as both $MT = PF = \Delta y$ and Δx approach zero as a limit we have

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{TC}{MT} = \lim_{\Delta y\to 0} \frac{FN}{PF} = \frac{\partial z}{\partial y}; \tag{5}$$

whence

$$\frac{TC}{MT} = \frac{\partial z}{\partial y} + e_2, \tag{6}$$

where $e_2 \to 0$ as $\Delta x \to 0$, $\Delta y \to 0$. This gives

$$TC = \left(\frac{\partial z}{\partial y} + e_2\right) \Delta y. \tag{7}$$

Substituting the values of the line-segments KM and TC in Δz, we get

$$(8) \qquad \Delta z = \frac{\partial z}{\partial x} \cdot \Delta x + \frac{\partial z}{\partial y} \cdot \Delta y + e_1 \cdot \Delta x + e_2 \cdot \Delta y.$$

This formula for the change in z due to changes in x and y is of no value for finding the change of a given function. Such a problem is merely an arithmetical calculation of the difference of the two values of the function.

However, if we observe that e_1 and e_2 depend upon Δx and Δy, we see that for sufficiently small values of Δx and Δy the two terms $e_1 \cdot \Delta x + e_2 \cdot \Delta y$ become very small. Hence the remaining two terms of Δz may be used to approximate the change in z.

These two terms are *defined* as the ***differential of z,*** and represent the ***approximate change*** or ***error in z*** for given changes or errors in x and y. That is,

$$(9) \qquad dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy,$$

where we write dx and dy for Δx and Δy as both are independent increments. This is a useful formula, as the partial derivatives and increments are readily combined to give approximate changes in the function of two variables.

As dx and dy are independent increments, we have

$$(10) \qquad d_x z = \frac{\partial z}{\partial x} dx, \qquad d_y z = \frac{\partial z}{\partial y} dy,$$

where $d_x z$ and $d_y z$ represent the ***partial differentials*** of z with respect to x and y respectively. We notice that dz becomes $d_x z$ when $dy = 0$, and $d_y z$ when $dx = 0$. Therefore *the differential of z is the sum of the partial differentials of z with respect to each x and y.*

EXAMPLES

1. If $z = x^2 - 2\,xy + y^2$, find dz, $d_x z$, $d_y z$.

SOLUTION. Since $\partial z/\partial x = 2\,x - 2\,y$ and $\partial z/\partial y = -2\,x + 2\,y$, we have $d_x z = 2(x - y)dx$ and $d_y z = 2(y - x)dy$. From these we obtain $dz = d_x z + d_y z = 2(x - y)dx + 2(y - x)dy$.

2. In measuring two sides of a triangle which include an angle of 30° one side is found to be 27 inches with a possible error of 0.10 inch, and the other

13 inches with a possible error of 0.05 inch. What is an approximate value for the largest possible error in the area of the triangle due to the errors in measuring the sides?

SOLUTION. The area A of the triangle may be given by the relation $A = (1/2)xy \sin 30° = (1/4)xy$. An approximate error in A due to errors in x and y is

$$dA = \frac{1}{4}(y\,dx + x\,dy).$$

We have given $x = 27$ in., $y = 13$ in., $dx = \pm 0.10$ in., and $dy = \pm 0.05$ in. Since the largest possible value for dA is desired, we assume dx and dy to have the same sign since their coefficients have the same sign. Substitution gives $dA = (1/4) \cdot 13 \cdot (1/10) + (1/4) \cdot 27 \cdot (1/20) = 53/80$ sq. in. The ***actual error*** is $\Delta A = A_2 - A_1$, where A_2 is computed by using $x = 27.1$ in. and $y = 13.05$ in., and A_1 by using $x = 27$ in., $y = 13$ in. Such computation being usually laborious, the value of dA is used as an approximation of the error ΔA.

The ***relative error*** is defined as $\Delta A/A$ and again the ***approximate relative error*** dA/A is generally requested. The ***percentage error*** is of course 100 times the relative error. It is convenient to take the logarithm of a function if the approximate relative error is desired. Thus for this example, we have

$$\log A = \log x + \log y - \log 4.$$

Then the differential gives

$$\frac{dA}{A} = \frac{dx}{x} + \frac{dy}{y} = \frac{0.10}{27} + \frac{0.05}{13} = 0.0075.$$

This shows that *an approximation for the relative error of a function is the sum of the approximations of the relative errors of each factor.* This expressed as a percentage is evidently $(3/4)\%$.

The expressions for Δz and dz are readily extended to the cases of three or more variables. Thus, if $u = f(x, y, z)$, we have

$$\Delta u = \frac{\partial u}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial y} \cdot \Delta y + \frac{\partial u}{\partial z} \cdot \Delta z + e_1 \cdot \Delta x + e_2 \cdot \Delta y + e_3 \cdot \Delta z,$$

by a line of reasoning that is essentially that given for two variables except for the geometric interpretation.

The differential of u is *defined* as

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy + \frac{\partial u}{\partial z} \cdot dz,$$

and is used to approximate the value of Δu just as dz is used to approximate Δz.

Exactly similar relations hold for four or more independent variables as long as the number of independent variables is finite.

PROBLEMS

Use the differential in evaluating the following unless exact results are called for.

1. If x differs from 4 units by at most 0.02 unit and y from 3 by 0.01 unit, approximate the difference between $x^2 + y^2$ and 25 sq. units.

Ans. 0.22 sq. unit.

2. A right circular cylinder of altitude 10 ft. and radius 4 ft. has its altitude increased 0.1 ft. and its radius decreased 0.01 ft. Approximate the change resulting in the volume.

3. Approximate the possible error in the area of an ellipse (πab sq. units) due to a 1% error in a and a 2% error in b. *Ans.* 3%.

4. If $z(y^2 - x^2) = xy$ and $x = 2$, $y = 3$, $\Delta x = 0.0016$, $\Delta y = 0.0123$, find (*a*) Δz; (*b*) dz; (*c*) the approximate relative change in z; (*d*) the percentage change in z.

5. Approximate the volume of a circular cylinder with radius 2.997 in. and altitude 10.02 in. *Ans.* $90\,\pi$ cu. in.

6. The hypotenuse and a leg of a right triangle are 5 in. and 4 in., respectively. If the hypotenuse is decreased 0.01 in. and the leg increased 0.01 in., approximate the change in the third side, the triangle being kept a right triangle.

7. The power consumed in an electrical resistor is given by $P = E^2/R$ watts. If $E = 100$ volts and $R = 5$ ohms, how does the power change if E is decreased 2 volts and R is decreased 0.3 ohm? *Ans.* $dP = 40$ watts.

8. If the length and width of a rectangle are measured as 3.5 and 2.3 ft. with 0.01 ft. possible error in each, approximate the possible error in the computed area.

9. Find the exact and approximate error in the calculated volume of a right circular cone if $r = 4.95$ in. and $h = 4.1$ in. but 5 in. and 4 in. are used.

Ans. 0.1534 cu. in.; $\pi/6$ cu. in.

10. If $S = LH^2/\sqrt{D}$ and L, H, D are measured as 4, 3, and 25 in. respectively with a possible error of 1% in each, approximate the possible percentage error in S.

11. Approximate the relative error in Problem 9. *Ans.* 1/200.

12. If $z = x^3 + y^3 - 3\,x^2y$, and $x = 2$, $y = 3$, $\Delta x = 0.01$, $\Delta y = 0.001$, find Δz and dz.

13. If x and y are measured as 5 and 3 units with possible errors of 0.02 unit in x and 0.01 in y, approximate the possible resulting error in $M = x\sqrt{x^2 - y^2}$. *Ans.* $\pm\, 97/400$.

14. The radius of a right circular cone is measured as 7 ft. and its slant height as 25 ft. with a possible error of 1% in each. What possible error results in the computed volume?

15. Two sides of a triangle are 6 in. and 8 in. long, respectively, and they include an angle of 30°. Approximate the change in the area if the shorter side is decreased 1/25 in. and the longer is increased by the same amount.

Ans. $-1/50$ sq. in.

16. Approximate the possible error in the triangle of Problem 15 if the sides were measured accurately to within 0.1 in.

17. Find the maximum percentage error allowable in measuring the dimensions of a circular cylinder so that the computed volume will be correct to within 1%. *Ans.* If $dr = dh$ then $100(dr/r) = h/(2h \pm r)$.

18. Find the maximum percentage error allowable in measuring the diameter of a sphere if the computed volume must be correct to within 1%.

19. The length L and the period P of a simple pendulum are connected by the law $4\pi^2 L = P^2 g$. If L is calculated for $P = 1$ sec., $g = 32$ ft./sec.², approximate the error in L if P is really 1.02 sec., $g = 32.01$ ft./sec.² Find the approximate percentage error also. *Ans.* $1.29/4\pi^2$ units, 129/32%.

20. Using the pendulum law of Problem 19 for possible errors of 0.1% and 0.2%, respectively, in L and g, approximate the possible percentage error in P.

21. Suppose L in Problem 19 is measured as 100 units with a possible error of 1/2 unit, and P as 2 units with a possible error of 0.01 unit. Approximate the possible percentage error in g. *Ans.* 1½%.

22. If $R = (V^2 \sin 2\theta)/32$, approximate the possible error in computing R for $V = 5000$ ft./sec. with a possible error of 10 ft./sec. and $\theta = \pi/3$ with a possible error of 3 min.

23. The altitude and diameter of a circular cylinder are measured as 10 in. and 6 in., respectively. If a 4% error is made in the diameter, what error in the altitude will counteract this error in the computed volume?

Ans. -0.8 in.

24. Approximate the resulting change in each of the following cases.

(*a*) $z = \log(x^2 + y^2)$ for $x = y = 1$ and $dx = \pm dy = \pm 0.01$.
(*b*) $q = e^{x/2} \sin 2y$ for $x = 2$, $y = \pi/3$ and $dx = \pm 0.01$, $dy = \pm 0.01$.
(*c*) $T = (\tan y)/(\log x)$, for $x = 2$, $y = \pi/4$, and $dx = \pm 0.1$, $dy = \pm 3'$.

25. The density D of a body is $W_1/(W_1 - W_2)$ where W_1 is its weight in air and W_2 its weight in water. If $W_1 = 1200$ gr. and $W_2 = 10$ gr., what is approximately the greatest percentage error in D as computed from these weights if there is a possible error of 0.50 gr. in W_1 and 0.01 gr. in W_2?

Ans. 0.00119%.

135. The Derivative of $z = f(x, y)$ with Respect to Some Other Variable. If it is desired to find the rate of change of z with respect to some other variable t, it is obviously necessary that x and y be functions of t. This makes z a function of t and the

derivative of z with respect to t is defined as in the case of a function of one variable.* Thus, if $x = \phi(t)$ and $y = \psi(t)$, we have

$$(1) \qquad \frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\partial z}{\partial x} \cdot \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y} \cdot \frac{\Delta y}{\Delta t} + e_1 \frac{\Delta x}{\Delta t} + e_2 \frac{\Delta y}{\Delta t} \right),$$

or

$$(2) \qquad \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt},$$

where Δx and Δy approach zero as a limit as $\Delta t \to 0$, and whence e_1 and $e_2 \to 0$. It is assumed, of course, that dx/dt and dy/dt exist.

Let $ABCD$ represent a portion of the surface $z = f(x, y)$.

Then the equations $x = \phi(t)$, $y = \psi(t)$ define a cylindrical surface NM. This cylinder intersects the surface in the curve NK and the xy plane in the curve LM. Hence dz/dt represents the rate of change of QP with respect to t as Q moves along the curve LM.

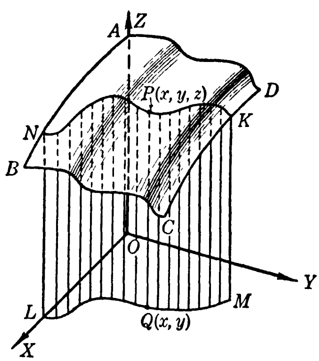


FIG. 141

A function u of three variables x, y, and z, each of which depends upon another variable t, has as its derivative the expression

$$(3) \qquad \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt},$$

where u, x, y, and z are differentiable functions of t.

If $z = f(x, y)$ and $y = F(x)$, so that z is a function of x alone, we may find the rate of change of z with respect to x as in (2). That is,

$$\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{\partial z}{\partial x} \cdot \frac{\Delta x}{\Delta x} + \frac{\partial z}{\partial y} \cdot \frac{\Delta y}{\Delta x} + e_1 \frac{\Delta x}{\Delta x} + e_2 \frac{\Delta y}{\Delta x} \right).$$

Then if Δy, e_1, and $e_2 \to 0$ when $\Delta x \to 0$ we have

$$(4) \qquad \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx},$$

if $\lim_{\Delta x \to 0} \Delta y/\Delta x$ exists.

* Of course x and y may not be functions of t alone, as $x = \phi(t, u)$, $y = \psi(t, u)$, and still the derivative of z with respect to t have the same derivation, provided $z = f(x, y)$ is still a function of t only.

Similarly, by considering z a function of y alone, we get

$$\frac{dz}{dy} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dy} + \frac{\partial z}{\partial y}. \tag{5}$$

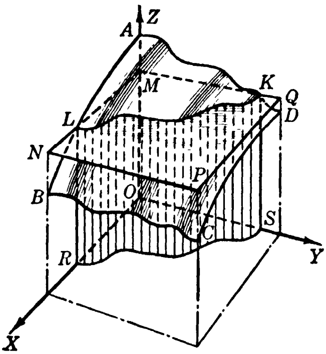


FIG. 142

These formulas (4) and (5) are evidently the same as (2) if x replaces t or y replaces t, respectively.

The plane $z = c$ intersects the surface $z = f(x, y)$ along the curve LK. If we let z be constant in relations (4) and (5), the left-hand member of each is zero and these become, respectively,

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = 0, \quad \frac{\partial z}{\partial x} \cdot \frac{dx}{dy} + \frac{\partial z}{\partial y} = 0.$$

Solving either of these, we obtain the relation

$$\frac{dy}{dx} = -\frac{\dfrac{\partial z}{\partial x}}{\dfrac{\partial z}{\partial y}}. \tag{6}$$

This gives the slope of the curve RS, which is the projection of the curve LK upon the xy plane.

If $z = f(x, y)$ is written as $F(x, y, z) = 0$, where z is given implicitly as a function of x and y, we find expressions for $\partial z/\partial x$ and $\partial z/\partial y$ by assuming first y constant and then x constant.

Thus, if y is constant, differentiation of $F(x, y, z) = 0$ with respect to x gives

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0,$$

where $\partial z/\partial x$ means dz/dx for y constant. Solving this for $\partial z/\partial x$, we find

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}, \quad \text{if} \quad \frac{\partial F}{\partial z} \neq 0. \tag{7}$$

Similarly,

$$(8) \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}, \qquad \text{if} \qquad \frac{\partial F}{\partial z} \neq 0.$$

EXAMPLES

1. If the radius of a right circular cone is increasing 3 inches per second and its height is decreasing 4 inches per second, how fast is its volume changing when $r = 6$ inches and $h = 12$ inches?

SOLUTION. The volume of the cone is $V = (1/3)\,\pi r^2 h$. Therefore

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt} = \frac{2}{3}\pi r h\cdot\frac{dr}{dt} + \frac{1}{3}\pi r^2\cdot\frac{dh}{dt}.$$

But $r = 6$ in., $h = 12$ in., $dr/dt = 3$ in./sec., and $dh/dt = -4$ in./sec., from the conditions given. Hence, substituting, we have

$$\frac{dV}{dt} = \frac{2}{3}\pi\cdot 6\cdot 12\cdot 3 + \frac{1}{3}\pi\cdot 36(-4) = 96\,\pi \text{ cu. in./sec.}$$

Since dV/dt is positive, the volume is increasing $96\,\pi$ cu. in./sec. at that instant.

2. If $z = \sin^{-1}[(1 + x)/(1 + y)]$ and $x = \sin t$, $y = \cos t$, find the rate of change of z with respect to t when $t = 0$.

SOLUTION. By differentiation, we find

$$\frac{dz}{dt} = \frac{1}{\sqrt{1 - \left(\dfrac{1+x}{1+y}\right)^2}}\cdot\frac{1}{1+y}\cdot\frac{dx}{dt} + \frac{1}{\sqrt{1 - \left(\dfrac{1+x}{1+y}\right)^2}}\cdot -\frac{1+x}{(1+y)^2}\cdot\frac{dy}{dt}.$$

But $dx/dt = \cos t$, $dy/dt = -\sin t$; so for $t = 0$ we have $x = 0$, $y = 1$, $dx/dt = 1$, $dy/dt = 0$. Substituting these values in the expression above, we obtain

$$\frac{dz}{dt} = \frac{1}{\sqrt{1 - \left(\dfrac{1}{1+1}\right)^2}}\left[\frac{1}{1+1}\cdot 1 - \frac{1}{(1+1)^2}\cdot 0\right] = \frac{1}{\sqrt{3}}.$$

PROBLEMS

1. The volume and radius of a cylindrical boiler are expanding at the rate of $0.8\,\pi$ cu. ft./min. and 0.002 ft./min., respectively. How fast is the length of the boiler changing when the volume is $20\,\pi$ cu. ft. and the radius is 2 ft.?

Ans. 0.19 ft./min.

2. The radius of a right circular cone is decreasing 2 in./sec. and its slant height is increasing 3 in./sec. How fast is the volume changing when $r = 4$ in., $h = 3$ in.?

3. Suppose that the x and y coordinates of a point on the surface $x^2 + 2y^2 - 3 + z = 0$ are changing 3 units/sec. and 4 units/sec., respectively. How is z changing at $(2, 3, -19)$? *Ans.* Decreasing 60 units/sec.

4. A particle at A on a line AB which is 28 ft. long starts toward B at 4 ft./min. and at the same time another particle leaves B in a direction which makes $60°$ with BA at 4 ft./min. How fast are the two particles approaching each other after 1 min.?

5. The top of a 20 ft. ladder leaning against a vertical wall slides down at the rate of 3 ft./sec. (*a*) How fast is the foot moving along the ground, which makes an angle of $2\pi/3$ with the wall, when the ladder makes an angle of $\pi/6$ with the ground? (*b*) At the same position how fast is the area of the triangle formed by ladder, wall, and ground changing?

Ans. 3 ft./sec.; no rate of change.

6. The legs of a right triangle at a given time are 2 ft. and 4 ft. and are increasing 1 ft./min. How fast is the area of the triangle changing? How fast is the perimeter changing?

7. In a right circular cone of altitude 4 ft. and radius 3 ft., the slant height is increasing 0.1 ft./sec., and the radius is decreasing 0.2 ft./sec. How fast is the volume changing? *Ans.* Decreasing 0.775π cu. ft./sec.

8. Find the rate of change of the area of a triangle if its shorter side is decreasing 0.05 in./sec. and its longer side is increasing 1/30 in./sec., when they are 6 in. and 8 in. respectively! The included angle is $\pi/6$.

9. A point on the surface $z = \log xy$ at the point $(2, 3, \log 6)$ is moving so that the x coordinate increases 1 unit/sec., and the y coordinate decreases 4 units/sec. How is z changing? *Ans.* $-5/6$ unit/sec.

10. Given $z = \tan^{-1}[x/(1+y)]$, $x = \cos t$, $y = \sin t$. Find (*a*) dz/dt at $t = \pi/4$; (*b*) dz/dx at $t = \pi/6$.

11. The radius and altitude of a right circular cone at the instant when $r = 12$ ft., $h = 5$ ft., are changing so that r is increasing 0.2 ft./sec. and the slant height decreasing 0.1 ft./sec. How fast is the total surface changing?

Ans. 6.2π sq. ft./sec.

12. Two sides and the included angle of a triangle are such that the shorter side is decreasing 0.1 ft./sec. and the longer side is increasing 0.15 ft./sec., the angle is decreasing 0.1 radian/sec. How fast is the area changing for the respective values 10 ft., 15 ft., $\pi/3$?

13. We move along a hill, a part of which may be represented by $z = 4 - x^2 - 2y^2$, in such a direction that x increases 2 units/sec., and y decreases 3 units/sec. How is z (the height) changing?

Ans. $dz/dt = 4(3y - x)$ units/sec.

14. If $e = (\sqrt{a^2 - b^2})/a$, find the rate of change of e when $a = 8$ in., $b = 6$ in. and each is changing at 0.5 in./sec.

15. Find the rate of change of an acute angle of a right triangle if the opposite and adjacent legs are 10 ft. and 12 ft., respectively, and are changing at the rate of 1 ft./min. and -2 ft./min. *Ans.* 8/61 rad./min.

16. Given any continuous function of three variables, $F(x, y, z) = 0$, show that

$$\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x} = -1.$$

17. Given Van der Waal's formula $(p + a/v^2)(v - b) = ct$, where a, b, and c are constants. Find $\partial p/\partial v$, $\partial v/\partial t$, and $\partial t/\partial p$. Check your results by using Problem 16.

136. The Directional Derivative of $z = f(x, y)$. Since the value of z is determined when x and y are known, z is fixed for each point of the xy plane. Then if the point $P(x, y)$ moves, in general, z changes value and therefore has a rate of change at each point of the plane.

Suppose we desire the rate of change of z for the point $P(x, y)$ if P moves in the direction PR. Let PR make an angle θ with the x axis. If we assume that P has moved to an adjacent point Q on PR, the change in z is its increment due to the corresponding increments Δx and Δy.

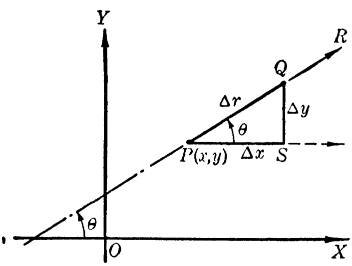


FIG. 143

The derivative of z for the point P in the direction PR is defined as the limit of the change of z divided by the distance Δr which P moves in the given direction as $\Delta r \to 0$. That is,

$$(1)\qquad \frac{dz}{dr} = \lim_{\Delta r \to 0}\frac{\Delta z}{\Delta r} = \lim_{\Delta r \to 0}\left(\frac{\partial z}{\partial x}\cdot\frac{\Delta x}{\Delta r} + \frac{\partial z}{\partial y}\cdot\frac{\Delta y}{\Delta r} + e_1\frac{\Delta x}{\Delta r} + e_2\frac{\Delta y}{\Delta r}\right).$$

Since Δx and $\Delta y \to 0$ as $\Delta r \to 0$ we have e_1 and e_2 approaching zero with Δr and therefore

$$(2)\qquad \frac{dz}{dr} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dr} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dr}.$$

But $dx/dr = \lim\limits_{\Delta r \to 0}(\Delta x/\Delta r)$, and $\Delta x/\Delta r = \cos\theta$, from triangle PQS. Also $Q \to P$ along the straight line QP and therefore the ratio $\Delta x/\Delta r$ is constant. Whence $dx/dr = \cos\theta$. Similarly,

$dy/dr = \sin\theta$. These values substituted in relation (2) give the ***directional derivative*** of z as

$$\frac{dz}{dr} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta. \tag{3}$$

The student must realize that this expression for dz/dr is entirely a function of θ when the point P has been chosen. That is, $\partial z/\partial x$ and $\partial z/\partial y$ are constants evaluated by means of the coordinates of P. Then, for a given θ, the direction PR is fixed and dz/dr represents the rate of change of PH as H moves along the curve HT. (Fig. 144.)

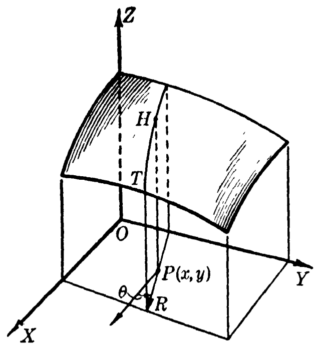


Fig. 144

Since dz/dr is a function of θ alone, we can find the θ which makes the directional derivative a maximum or minimum for any given point. Notice that θ is between 0° and 360° and is measured from the direction of the positive x axis.

EXAMPLES

1. If $z = x^2 + y^2$, find the rate of change of z for the point (3, 4) in the direction of the point (2, 6). What direction makes the rate of change a maximum?

Solution. $dz/dr = 2\,x\cos\theta + 2\,y\sin\theta$, and at (3, 4) this becomes

$$\frac{dz}{dr} = 6\cos\theta + 8\sin\theta.$$

Now θ, as shown in Fig. 145, is the second-quadrant angle which has its cosine equal to $-1/\sqrt{5}$ and its sine equal to $2/\sqrt{5}$. It follows that we have $dz/dr = -6/\sqrt{5} + 16/\sqrt{5} = 2\sqrt{5}$.

To find the θ which makes dz/dr a maximum, we have, as the first necessary condition,

$$\frac{d}{d\theta}\left(\frac{dz}{dr}\right) = -6\sin\theta + 8\cos\theta = 0,$$

whence $\tan\theta = 4/3$. This means that θ is either an acute angle or a third-quadrant angle. To determine which, we need the second derivative of dz/dr with respect to θ. This is

$$\frac{d^2}{d\theta^2}\left(\frac{dz}{dr}\right) = -6\cos\theta - 8\sin\theta.$$

This second derivative is negative for positive $\sin\theta$ and $\cos\theta$ and therefore dz/dr is a maximum if $\theta = \tan^{-1}(4/3)$ in the first quadrant. Since $\sin\theta$ and $\cos\theta$ are negative for θ in the third quadrant, we have

$$\frac{d^2}{d\theta^2}\left(\frac{dz}{dr}\right) > 0$$

for the $\tan^{-1}(4/3)$ in the third quadrant and hence dz/dr is a minimum in that direction.

If $\theta = \tan^{-1}(4/3)$, then $\cos\theta = \pm 3/5$, $\sin\theta = \pm 4/5$. Hence for the acute angle $dz/dr = 6(3/5) + 8(4/5) = 10$, and for the third-quadrant angle dz/dr is -10. We point out that the numerical values of dz/dr are the same in the two directions. This means that the extreme values of dz/dr are the same except that for one direction z is increasing and for the *opposite direction* z is decreasing.

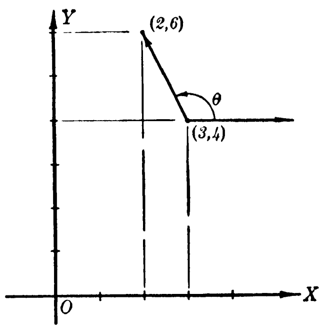


FIG. 145

2. If the temperature T at any point of the xy plane is given by $T = k/(x^2 + y^2)$, find the rate of change of the temperature at the point (3, 4) in the direction such that $\theta = 120°$.

SOLUTION. Since $T = k/(x^2 + y^2)$, we have

$$\frac{dT}{dr} = -\frac{2\,kx}{(x^2 + y^2)^2}\cos\theta - \frac{2\,ky}{(x^2 + y^2)^2}\sin\theta.$$

At the point (3, 4) for $\theta = 120°$, or $\cos\theta = -1/2$, $\sin\theta = \sqrt{3}/2$, this becomes

$$\frac{dT}{dr} = \frac{3\,k}{625} - \frac{4\,k\sqrt{3}}{625} = -0.006\,k.$$

PROBLEMS

Find the directional derivative for each function defined below. (Nos. 1–8.)

1. $z = 1/(x^2 + y^2)$, at (2, 3) toward $(-1, -1)$. *Ans.* $36/[5(13)^2]$.

2. $T = y^2 \tan^2 x$, at $(\pi/3, 2)$, $\theta = 5\,\pi/6$.

3. $Q = \log\sqrt{x^2 - 2\,y^2}$, at $(-2, 1)$ toward $(-6, -2)$. *Ans.* 7/5.

4. $z = \tan^{-1}(y/x)$, at (1, 1) toward $(-2, -3)$.

5. $z = \tan^{-1}[(1 - x)/y] + \tan^{-1}[(1 + x)/y]$, at (1, 1) toward the origin. *Ans.* $6/(5\sqrt{2})$.

6. $V = \log\,(x^2 + y^2)^{1/3}$, at (a, b) toward the origin.

7. $z = e^{-ax}\cos ay$, at $[0, \pi/(4\,a)]$ toward $[-\pi/(4\,a), 0]$. *Ans.* a.

8. $z = e^{-y^2}\cos x$, at $(\pi, 1)$ toward the origin.

Find extreme values for the directional derivatives desired in each of the following cases. (Nos. 9–18.)

9. $z = \log [(x^2 + y^2)/(x^2 - y^2)]$ at $(2, -1)$. *Ans.* Max. for $\theta_3 = \tan^{-1} 2$.

10. $z = e^{x^2/2y}$ at $(2, 1)$.

11. $z = \tan^{-1}(y/x)$ at $(5, 1)$. *Ans.* Max. for $\theta_2 = \tan^{-1}(-5)$.

12. $T = (x^2 + y^2)\tan^{-1}(y/x)$ at $(1, 1)$.

13. $z = \log(xy^2 - x^2y)$ at $(1, -1)$. *Ans.* Max. for $\theta_4 = \tan^{-1}(-1)$.

14. $z = e^x \tan^{-1} y$ at $(0, 1)$.

15. $z = e^{-ax} \sin ay$ at $[0, \pi/(4a)]$. *Ans.* Min. for $\theta = 7\pi/4$.

16. $z = y^2 \tan^2 x$ at $(\pi/3, 2)$.

17. $z = \log(x^2 + y^2)\sin(x + y)$ at $(\pi/4, \pi/4)$. *Ans.* Max. for $\theta = \pi/4$.

18. $u = e^{-y}\sin x + (1/3)e^{-3y}\sin 3x$ at $(\pi/3, 0)$.

19. The electrical potential at a point is given by $P = \log\sqrt{x^2 + y^2}$. What is its rate of change at $(0, 4)$ toward $(3, 0)$? In what direction from $(0, 4)$ is its rate of change a maximum? *Ans.* $-1/5$, $\pi/2$.

20. In what direction from $(3, 1)$ does the function $e^{2x}\tan^{-1}[x/(3y)]$ have zero rate of change?

21. In what direction from $(\frac{1}{2}, 1, 2\frac{3}{4})$ is the surface $z = 4 - x^2 - y^2$ rising most rapidly? *Ans.* $\tan^{-1} 2$, in third quadrant.

22. Use Problem 19 to find
(*a*) dP/dr at $(2, 3)$ in the direction of the x axis.
(*b*) dP/dr at $(2, 3)$ in the direction which makes $\pi/4$ with the y axis.
(*c*) dP/dr at $(2, 3)$ toward $(5, -1)$.
(*d*) dP/dr at $(2, 3)$ to be maximum.

23. Show that the directional derivative of $\log(x + \sqrt{x^2 + y^2})$ at any point P toward the origin O is numerically the same as the reciprocal of PO.

24. A portion of a hill may be represented by $4x^2 + y^2 + z - 9 = 0$. (*a*) Sketch the surface with the z axis directed upward. (*b*) What grade has a road through the point $(1, 2, 1)$ with $\theta = \pi/4$? (*c*) What direction has a contour line at $(1, 2, 1)$? (*d*) What direction at $(1, 2, 1)$ is the steepest? (*e*) What direction has a 2% grade at $(1, 2, 1)$?

25. On a hill represented by $z = 6 - 2x^2 - y^2$ find the direction of the contour line at $(1, 1, 3)$. Also the direction of steepest grade.
Ans. $\tan^{-1}(-2)$; $\tan^{-1}(1/2)$, in third quadrant.

26. At any point of the surface $z + r = 0$ what direction has no rise or fall; in what direction is the fall most rapid; what direction has a 3% rise?

137. Maxima and Minima of a Function $f(x, y)$. A function of two independent variables has a maximum value for the values $x = x_1$, $y = y_1$ provided that, for all values of Δx and Δy that are sufficiently small numerically, but not both zero, the quantity

$f(x_1, y_1)$ is greater than $f(x_1 + \Delta x, y_1 + \Delta y)$. For a minimum, $f(x_1, y_1)$ must be less than $f(x_1 + \Delta x, y_1 + \Delta y)$.

Geometrically, the function $z = f(x, y)$ represents some surface, and if $z_1 = f(x_1, y_1)$ is a maximum or minimum value of the function, the tangent plane to the surface at the corresponding point P_1 will be parallel to the xy plane. That is, its equation will be $z - z_1 = 0$. The equation of the tangent plane at the point P_1 is (§ 133):

$$z - z_1 = \frac{\partial z_1}{\partial x_1}(x - x_1) + \frac{\partial z_1}{\partial y_1}(y - y_1).$$

Now, if P_1 is a high point or a low point, these two equations must be identical; hence

$$(1) \qquad \frac{\partial z_1}{\partial x_1} = 0, \qquad \frac{\partial z_1}{\partial y_1} = 0.$$

These conditions are necessary but are not sufficient. Each derivative should be tested on both sides of the critical point P_1. Complete tests which are sufficient are discussed in more advanced courses.

PROBLEMS

Find the maximum or minimum values of each of the following functions of two variables. (Nos. 1–6.)

1. $f(x, y) = x^2 - 3x + 4y^2 + 4y + 1.$ *Ans.* $-9/4$ (min.).

2. $\phi(r, s) = r^2 + 4r + 2s^2 - s + 5.$

3. $f(z, t) = 5 + 6z - 4z^2 - 3t^2.$ *Ans.* $11/4$ (max.).

4. $f(x, z) = 1 + 6x - 9z - 2x^2 - 2z^2.$

5. $\phi(r, s) = r^2 - 3rs + 3s^2 + 4r - 10s + 6.$ *Ans.* $-10/3$ (min.).

6. $f(x, y) = 3x^2 + 2xy + 3y^2 - 6x + 4y.$

Find the high point or the low point of each of the following quadric surfaces. (Nos. 7–11.)

7. $z = 4 - (x - 2)^2 - (y + 3)^2.$ *Ans.* $(2, -3, 4)$ (high).

8. $z = 2x^2 - 3x + 3y^2 - 6y + 1.$

9. $z = 2x^2 - 6xy + 5y^2 - x + 3y + 2.$ *Ans.* $(-2, -3/2, 3/4)$ (low).

10. $z = 3x^2 + 5xy + 4y^2 - 7x - 2y + 6.$

11. $z = x^2 - 2y^2 - 3x + 5y - 1.$ *Ans.* Neither max. nor min.*

* The point $(3/2, 5/4, -1/8)$ satisfies the conditions $z_x = 0$, $z_y = 0$ but the section made by the plane $x = 3/2$ is concave upward at this point, while the section made by the plane $y = 5/4$ is concave downward.

12. What should be the dimensions of a rectangular open tank with a fixed surface area so as to have a maximum capacity?

13. Divide a number N into three parts such that their product shall be the greatest possible. *Ans.* $N/3$, $N/3$, $N/3$.

14. What dimensions should an open rectangular tank of given volume have in order that its inner surface shall be a minimum?

138. Envelopes. The curves given by $f(x, y, c) = 0$ for all values of c constitute a one-parameter family of curves. An illustration of such a family of curves is the different paths of projectiles fired with the same muzzle velocity in the same vertical plane but at all angles of elevation from 0° to 180°.

Two curves of the family, obtained by assigning to c the values c and $c + \Delta c$, may intersect in one or more points. As $\Delta c \to 0$ each intersection approaches a corresponding limiting position P for each value of c. The locus of the points P, if they exist, as c varies is called the ***envelope*** of the one-parameter family of curves.

Since P is the limiting position of the intersection of the curves $f(x, y, c) = 0$ and $f(x, y, c + \Delta c) = 0$ as $\Delta c \to 0$, its coordinates satisfy both

$$(1) \qquad f(x, y, c) = 0,$$

and

$$(2) \qquad \lim_{\Delta c \to 0} \frac{f(x, y, c + \Delta c) - f(x, y, c)}{\Delta c} \equiv \frac{\partial f(x, y, c)}{\partial c} = 0.$$

Hence equations (1) and (2) may be considered as parametric equations of the locus of P.

To obtain the rectangular equation of the envelope, eliminate the parameter c between these two equations.

If equations (1) and (2) are solved for x and y, the envelope may be written in the form

$$(3) \qquad \begin{cases} x = \phi(c), \\ y = \psi(c). \end{cases}$$

Hence the slope of the envelope is

$$(4) \qquad \frac{dy}{dx} = \frac{\dfrac{d\psi(c)}{dc}}{\dfrac{d\phi(c)}{dc}} = \frac{d\psi}{d\phi}.$$

The slope of a curve of the family (1) for a definite value of c is given by the equation

$$\frac{\partial f(x, y, c)}{\partial x} + \frac{\partial f(x, y, c)}{\partial y} \cdot \frac{dy}{dx} = 0. \tag{5}$$

But for any point (x, y) of the family (1)

$$df(x, y, c) = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy + \frac{\partial f}{\partial c} \cdot dc = 0. \tag{6}$$

If we also impose the condition that (x, y) lie on the envelope, we have, from (2) and (3),

$$\frac{\partial f}{\partial c} = 0, \qquad dx = d\phi, \qquad dy = d\psi.$$

Therefore (6) becomes

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{d\psi}{d\phi} = 0, \tag{7}$$

whence, from (4) and (5), the envelope and any member of the family of curves have the same slope at their intersection. This proves that *each member of the family of curves is tangent to the envelope of the family.*

EXAMPLE

Find the envelope of the family of lines $y = mx + p/m$ where p is a constant. Show that the envelope is tangent to each line of the family.

SOLUTION. Here the function is

$$f(x, y, m) = y - mx - \frac{p}{m} = 0,$$

therefore

$$\frac{\partial f}{\partial m} = -x + \frac{p}{m^2} = 0.$$

This last equation gives $m = \pm\sqrt{p/x}$.

Substituting in the first equation and simplifying, we have the parabola

$$y = \pm 2\sqrt{px}, \qquad \text{or} \qquad y^2 = 4\,px,$$

as the envelope of the family of lines.

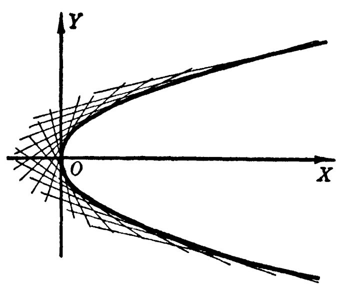


FIG. 146

Figure 146 shows the envelope and some of the lines of the family.

To show that the envelope is tangent to the lines, we eliminate y between

the equation of the envelope and that of the lines. This gives

$$m^2x^2 + 2\,px + \frac{p^2}{m^2} = 4\,px, \quad \text{or} \quad m^2x^2 - 2\,px + \frac{p^2}{m^2} = 0.$$

The left-hand side of this last equation is a perfect square; hence the two points where any one of the lines meets the parabola have the same abscissa, or the lines are tangent to the envelope. This method of showing tangency is suggested rather than finding the slopes of the curves and comparing them at intersections.

PROBLEMS

Find the envelope of each of the following families of curves, name the envelope, and show that the tangency condition is satisfied. (Nos. 1–14.)

1. $y = m^2x + 2\,m$. *Ans.* $xy + 1 = 0$.

2. $x \cos\theta + y \sin\theta = p$, θ the parameter.

3. $6\,x \cos m + 4\,y \sin m = 24$. *Ans.* $x^2/16 + y^2/36 = 1$.

4. $y = mx \pm \sqrt{a^2m^2 + b^2}$.

5. $x^2 + (y - k)^2 = 4$. *Ans.* $x = \pm 2$.

6. $y = px + 1 + 4/p$.

7. $y = mx - am^2$. *Ans.* $4\,ay = x^2$.

8. $2\,mx + m^2y = 5$.

9. $x^2 + (y - k)^2 = k^2$. *Ans.* $y = 0$.

10. $y = kx^2 + 1/k$.

11. $y = m^2x/2 + 1/cm$. *Ans.* $8\,c^2y^3 = 27\,x$.

12. $y^2 = p^3 - 3\,px$.

13. $(x - k)^2 + (y - k)^2 = 4\,k$. *Ans.* $(x - y)^2 = 4(x + y + 1)$

14. $(x - a)^2 + y^2 = 2\,ap - p^2$, a the parameter.

15. Find the envelope of the hypotenuse of a right triangle of constant area c. *Ans.* $2\,xy = \pm c$.

16. If the sum of the major and minor axes of a family of ellipses is constant, what is the envelope of the family?

17. Find the envelope of a family of lines the sum of whose intercepts on the coordinate axes is a constant k. *Ans.* $(x + y - k)^2 = 4\,xy$.

18. Find the envelope of the normals of the parabola $y^2 = 2\,px$.

19. Find the envelope of the circles through the origin with centers on $x^2 = 2\,y$. *Ans.* $x^2(y + 1) + y^3 = 0$.

20. A line-segment of fixed length moves with its extremities on the coordinate axes. Find the envelope.

CHAPTER X

INTEGRATION

139. Integration. Notation. Up to this point we have been interested in the rate of change of a given function with respect to its variables. However, it is desirable to be able to reverse the process of differentiation in order to find the function whose rate of change is given. For instance, we may want to know the distance a moving body has traveled in a given time when its velocity is known; or to find the equation of a curve when its slope is given. This inverse of the operation of differentiation is called ***integration.***

The inverse problem then is: *Given the derivative of an unknown function of one independent variable with respect to that variable, to find the function.* Familiarity with the formulas of differential calculus should enable us to write down at once many functions which have given rates of change.

THEOREM. *If two functions have the same derivative, they can differ only by an additive constant.*

PROOF. Let $f(x)$ and $g(x)$ be two functions which have the same derivative and let

$$\phi(x) = f(x) - g(x).$$

Then, by hypothesis,

$$\phi'(x) = f'(x) - g'(x) = 0.$$

That is, the rate of change of $\phi(x)$ with respect to x is zero for all values of x; hence $\phi(x)$ is a constant.

If the rate of change of a function y with respect to its variable x is given by $f(x)$, that is, if $dy = f(x)dx$ then the function

$$y = F(x) + C,$$

whose derivative is $f(x)$ is called an ***indefinite integral*** of $f(x)$. This relation between $f(x)$ and $F(x) + C$ is denoted by the symbol

$$\int f(x)dx = F(x) + C, \tag{1}$$

and is read: *The integral of* $f(x)dx$ *is* $F(x) + C$. Hence

$$d[F(x) + C] = f(x)dx,$$

and

$$\int f(x)dx = F(x) + C$$

indicate inverse operations.

The function $f(x)$ is called the ***integrand.*** The constant C is called the ***constant of integration,*** and may have any value whatever. It is determined in each problem by means of information in addition to the rate of change of the unknown function.

The expressions $\int au^n du$, $\int a\,du$, $\int a\,du/u$ may also be written $a\int u^n du$, $a\int du$, $a\int du/u$ because

$$d\left(\frac{au^{n+1}}{n+1}\right) = a \cdot d\left(\frac{u^{n+1}}{n+1}\right), \qquad d(au) = a \cdot du,$$

and $d(a \log u) = a \cdot d(\log u)$. *Thus a constant factor may be removed from the integrand and placed before the integral sign.* This operation is usually very desirable, as the simplified integrand is more easily handled during the process of integration.

From the forms above, we have the formulas:

$$\text{(I)} \qquad \int au^n du = a\int u^n du = \frac{au^{n+1}}{n+1} + C, \qquad n \neq -1.$$

$$\text{(II)} \qquad \int a\,du = a\int du = au + C.$$

When $n = -1$, the expression under the integral sign in formula (I) becomes $a\,du/u$ and so we have a third integral formula,

$$\text{(III)} \qquad \int a\frac{du}{u} = a \log u + C, \qquad u > 0.$$

Since the differential of the sum of a finite number of terms like au^n is the sum of their differentials, we have

$$\text{(IV)} \qquad \int [f_1(u) + f_2(u) + \cdots + f_n(u)]du$$
$$= \int f_1(u)du + \int f_2(u)du + \cdots + \int f_n(u)du$$
$$= F_1(u) + F_2(u) + \cdots + F_n(u) + C,$$

if each $f_i(u)\,du$ is $dF_i(u)$ and if $i = 1, 2, \cdots n$, where n is finite.

EXAMPLES

1. Find $\int 5\,dx$.

SOLUTION. Formula (II) gives $\int 5\,dx = 5\int dx = 5x + C$.

2. Find $\int (x^3 + 2x^2 - 7)\,dx$.

SOLUTION. Formulas (I), (II), and (IV) give

$$\int (x^3 + 2x^2 - 7)\,dx = \frac{x^4}{4} + \frac{2x^3}{3} - 7x + C.$$

3. Find $\int 2(x^2 + a^2)^2\,x\,dx$.

SOLUTION. Expanding the parenthesis, we have

$$\int 2(x^2 + a^2)^2\,x\,dx = 2\int (x^5 + 2a^2x^3 + a^4x)dx$$

$$= \frac{x^6}{3} + a^2x^4 + a^4x^2 + C.$$

However, a result is more readily obtained if the student recognizes that $2x\,dx = d(x^2 + a^2)$. In that case the integral is seen to be in the form of $\int au^n du$ where $a = 1$, $u = x^2 + a^2$, $du = 2x\,dx$ and $n = 2$. This makes it possible to write

$$\int (x^2 + a^2)^2\,2x\,dx = \frac{(x^2 + a^2)^3}{3} + C.$$

In these two seemingly different results, terms involving x occurring in the one are identical with those in the other; the results differ at most by an *additive constant*. This merely means that the constant of integration C does not represent the same quantity in both cases.

4. Find $\int x^3\,dx/(x - 3)$.

SOLUTION. When the integrand is an improper algebraic fraction, it should be reduced to a mixed number before integration. Thus,

$$\int \frac{x^3}{x - 3}dx = \int \left(x^2 + 3x + 9 + \frac{27}{x - 3}\right)dx$$

$$= \frac{x^3}{3} + \frac{3x^2}{2} + 9x + 27\log(x - 3) + C.$$

Here we use formula (III) to integrate the fractional part of the integrand. Evidently, $a = 27$, $u = x - 3$, and $du = dx$.

PROBLEMS

Integrate each of the following expressions.

1. $\int (1 - x - x^2)dx$. *Ans.* $x - x^2/2 - x^3/3 + C$.

2. $\int (10 - 7t + t^2)dt$.

3. $\int (1/2)(t^3/3 - 3t^2 + 5t)dt$.

Ans. $(1/2)(t^4/12 - t^3 + 5t^2/2) + C$.

4. $\int (1 + 2/x^2 + 3/x^3)dx.$

5. $\int (\sqrt{gt} - \sqrt[3]{V_0t^2})dt.$ *Ans.* $(2/3)g^{1/2}t^{3/2} - (3/5)V_0^{1/3}t^{5/3} + C.$

6. $\int (ax^{2/3} - bx^{-2/3})dx.$

7. $\int (2\sqrt{t} + 3\sqrt[3]{t} + 4\sqrt[4]{t})dt.$

Ans. $(4/3)t^{3/2} + (9/4)t^{4/3} + (16/5)t^{5/4} + C.$

8. $\int \left(\sqrt[3]{x^4} + \frac{1}{\sqrt[3]{x}}\right) dx.$

9. $\int \frac{4 - x^3}{x^4} dx.$ *Ans.* $-(4/3)x^{-3} - \log x + C.$

10. $\int (y/4 - 3)^2 dy.$

11. $\int (a^{2/3} - x^{2/3})^2 dx.$ *Ans.* $a^{4/3}x - (6/5)a^{2/3}x^{5/3} + (3/7)x^{7/3} + C.$

12. $\int \frac{x^2}{x - 1} dx.$

13. $\int \frac{3x^3}{1 - x} dx.$ *Ans.* $-x^3 - 3x^2/2 - 3x - 3\log(x - 1) + C.$

14. $\int \frac{x^2 + x + 1}{x + 1} dx.$

15. $\int \frac{6x^2 + x}{3x + 2} dx.$ *Ans.* $x^2 - x + (2/3)\log(3x + 2) + C.$

16. $\int \frac{6x^2 - 14x}{3x - 1} dx.$

140. The Integral of a Power of a Function. Due to the great importance of the integration formulas given in the preceding article, we shall add a group of problems which are somewhat different from those of that article. There we used the formulas involving u^n, but in each problem u was represented by a single letter. We now stress the fact that u *may represent any function of a variable and consequently that du is then the differential of that function and not the differential of the original independent variable.*

Failure to notice and understand this causes the beginner in integration a great amount of trouble. Hence, in applying formulas (I), (II), (III), and (IV), the student must examine the form $f(x)dx$ in $\int f(x)dx$ for the purpose of separating it into two parts, one of which shall represent u^n, and the other du, of the formulas. In choosing the function which may represent u, such natural groupings as quantities under radical signs, or in

parentheses, or denominators of fractions, should be considered. It is of utmost importance, however, that, having chosen the group of symbols to represent u^n, the remaining part of the expression under the integral sign shall differ at most by a constant factor from du.

EXAMPLES

1. Integrate $\int x^2(x^3 + 3)^{1/3}\,dx$.

Solution. Since $x^3 + 3$ is a radicand, we examine the remaining factors of the expression to see if they form the differential of $x^3 + 3$ to within a constant factor. That is, setting $u = x^3 + 3$, we get $du = 3\,x^2\,dx$. Hence the remaining factors $x^2 dx$ are equal to $du/3$. Now substituting these values for $(x^3 + 3)^{1/3}$ and $x^2\,dx$, we may write the original integral $\int x^2(x^3 + 3)^{1/3}\,dx$ in the form $\int u^{1/3}\cdot du/3$, or $(1/3)\int u^{1/3}\cdot du$. Integrating, we have

$$\begin{aligned}\int x^2(x^3 + 3)^{1/3}\,dx &= \frac{1}{3}\int u^{1/3}\,du\\ &= \frac{u^{4/3}}{4} + C = \frac{1}{4}(x^3 + 3)^{4/3} + C.\end{aligned}$$

If the student can see what constant factor is needed to change the given integral into the form of a formula, he can multiply and divide by this constant and a change of variable is not necessary. Thus, we may write

$$\begin{aligned}\int x^2(x^3 + 3)^{1/3}\,dx &= \frac{1}{3}\int (x^3 + 3)^{1/3}\cdot(3\,x^2\,dx)\\ &= \frac{(x^3 + 3)^{4/3}}{4} + C,\end{aligned}$$

directly without explicitly using the variable u. However, even then u is used implicitly.

2. Find $\displaystyle\int \frac{\sin\theta}{1 + \cos\theta}\,d\theta$.

Solution. Any fractional integrand should make the student recall the expression du/u. Hence we shall try setting $u = 1 + \cos\theta$, whence $du = -\sin\theta\,d\theta$, and substitution gives

$$\begin{aligned}\int \frac{\sin\theta}{1 + \cos\theta}\,d\theta = -\int\frac{du}{u} &= -\log(u) + C\\ &= -\log(1 + \cos\theta) + C.\end{aligned}$$

Without the explicit use of u, this is readily written

$$\int \frac{\sin\theta}{1 + \cos\theta}\,d\theta = -\int\frac{d(1 + \cos\theta)}{1 + \cos\theta} = -\log(1 + \cos\theta) + C.$$

3. Find $\int \sec^4\theta\,d\theta$.

Solution. An integrand which is trigonometric should be examined for a factor which is the derivative of some one of the trigonometric functions.

If such a factor is present and the remaining factors of the integrand can be expressed in terms of the trigonometric function whose derivative is present, the integral can readily be put into the form of one of the formulas.

In this example, $\sec^2 \theta \, d\theta$ is the differential of $\tan \theta$; hence we shall express the remainder of the integrand in terms of $u = \tan \theta$. This is readily possible as the remaining factor is $\sec^2 \theta$, which is equal to $1 + \tan^2 \theta$. Substitution gives

$$\int \sec^4 \theta \, d\theta = \int (1 + u^2) \, du = u + \frac{u^3}{3} + C.$$
$$= \tan \theta + \frac{\tan^3 \theta}{3} + C.$$

PROBLEMS

Integrate each of the following expressions by means of suitable substitutions for u and du.

1. $\int \sqrt{1 + x} \, dx.$ *Ans.* $(2/3)(1 + x)^{3/2} + C.$
2. $\int y(a + by^2)^{-2} \, dy.$ $-\frac{1}{2b(a+by^2)} + C$
3. $\int x^2 \sqrt[3]{x^3 - 3} \, dx.$ *Ans.* $(1/4)(x^3 - 3)^{4/3} + C.$
4. $\int x\sqrt{x^2 - 1} \, dx.$ $\frac{(x^2-1)^{3/2}}{3} + C$
5. $\int \frac{x + 2}{x^2 + 4x} \, dx.$ *Ans.* $(1/2) \log (x^2 + 4x) + C.$
6. $\int \frac{\cos \theta}{1 + \sin \theta} \, d\theta.$
7. $\int \frac{\sec^2 \theta}{2 \tan \theta - 3} \, d\theta.$ *Ans.* $(1/2) \log (2 \tan \theta - 3) + C.$
8. $\int \sin \theta \cos \theta \, d\theta.$ $\frac{\sin^2 \theta}{2} + C$
9. $\int \sec^2 \theta \tan \theta \, d\theta.$ *Ans.* $(1/2) \tan^2 \theta + C.$
10. $\int \cos \theta \, (1 - \sin^2 \theta) \, d\theta.$
11. $\int \frac{x \, dx}{(x - 1)^{3/2}}$ *Ans.* $2[\sqrt{x - 1} - 1/\sqrt{x - 1}] + C.$
12. $\int \frac{\sec^2 3\theta}{(1 - \tan 3\theta)^{1/3}} \, d\theta.$ $-\frac{1}{2}(1 - \tan 3\theta)^{2/3}$
13. $\int x \operatorname{ctn} x^2 \csc x^2 \, dx.$ *Ans.* $-(1/2) \csc x^2 + C.$
14. $\int \sin^3 2\theta \, d\theta.$ $-\frac{\cos 2\theta}{2} -$
15. $\int x^2(3 - 2x^3)^{2/3} \, dx.$ *Ans.* $-(1/10)(3 - 2x^3)^{5/3} + C.$

16. $\int 4x\sqrt{x^2+8}\,dx.$ $\frac{4}{3}(x^2+8)^{3/2}+$

17. $\int \frac{x+3}{\sqrt{x^2+6x}}\,dx.$ *Ans.* $\sqrt{x^2+6x}+C.$

18. $\int \sqrt{\frac{a^{1/2}+x^{1/2}}{x}}\,dx.$ $\frac{4}{3}(\sqrt{a}+\sqrt{x})^{3/}$

19. $\int \cos\theta\sqrt{1-\sin\theta}\,d\theta.$ *Ans.* $-(2/3)\sqrt{(1-\sin\theta)^3}+C.$

20. $\int \frac{\sin 2x}{(3-4\cos 2x)^{1/2}}\,dx.$ $= \frac{1}{8}\log(3-4\cos 2x)$

141. Integrals of Exponential and Trigonometric Functions. Additional integration formulas are obtained by the inverse of certain known formulas of differentiation. Thus

$$\text{(V)} \qquad \int a^u\,du = \frac{a^u}{\log a} + C.$$

$$\text{(VI)} \qquad \int e^u\,du = e^u + C.$$

In these formulas we must realize that the differential factor du is the differential of u which represents any function of one variable and occurs as the *exponent only* of a or e.

EXAMPLES

1. Evaluate $\int 3\,xe^{x^2-1}\,dx$.

SOLUTION. The u of formulas (V) and (VI) appears as the exponent of a or e; hence we set $u = x^2 - 1$. Then $du = 2\,x\,dx$ or $x\,dx = du/2$. Substitution gives

$$\int 3\,xe^{x^2-1}\,dx = 3\int e^u\frac{du}{2} = \frac{3}{2}\int e^u du = \frac{3}{2}e^{x^2-1} + C.$$

2. Evaluate $\int 8^{1-x}\,dx$.

SOLUTION. Here we must set $u = 1 - x$, whence $du = -dx$ and using formula (V), we have

$$\int 8^{1-x}dx = \int 8^u(-du) = -\int 8^u\,du = \frac{-8^{1-x}}{\log 8} + C.$$

3. Evaluate $\int \frac{3\,e^{2ax}+1}{e^{2ax}-1}\,dx.$

SOLUTION. Since the derivative of an exponential function contains the same function, we are able to use the formula for $\int du/u$ only if the numerator has the same exponential forms as the denominator. For that reason exponential fractions as integrands are not treated in the same manner as algebraic

fractions. We proceed as follows. Since the numerator of the integrand is not the derivative of the denominator, we *start* to reduce the fraction to a mixed expression. To do this, write the integrand in the form

$$\frac{1 + 3\,e^{2ax}}{-1 + e^{2ax}},$$

and divide the denominator into the numerator *once* so that we have

$$\frac{1 + 3\,e^{2ax}}{-1 + e^{2ax}} = -1 + \frac{4\,e^{2ax}}{-1 + e^{2ax}}.$$

Then

$$\int \frac{3\,e^{2ax} + 1}{e^{2ax} - 1}\,dx = \int \left(-1 + \frac{4\,e^{2ax}}{e^{2ax} - 1}\right) dx$$

$$= -x + \frac{2}{a} \log\,(e^{2ax} - 1) + C,$$

because the numerator is $2/a$ times the derivative of the denominator of the fractional part of the integrand. This part may be changed into terms of u where u is set equal to the denominator. It is well actually to make such substitutions until the work becomes very familiar.

The two things to notice carefully in this solution are: first, *we arrange the members of the fraction in ascending powers of e;* and second, *we do not continue the division after the remainder becomes the same, except for a constant factor, as the derivative of the denominator.*

From the derivatives of the trigonometric functions we have the inverse formulas

(VII) $$\int \sin u\,du = -\cos u + C.$$

(VIII) $$\int \cos u\,du = \sin u + C.$$

(IX) $$\int \sec^2 u\,du = \tan u + C.$$

(X) $$\int \csc^2 u\,du = -\operatorname{ctn} u + C.$$

(XI) $$\int \sec u \tan u\,du = \sec u + C.$$

(XII) $$\int \csc u \operatorname{ctn} u\,du = -\csc u + C.$$

In these formulas the student must observe that u represents a function of some variable, say x or θ, and du is the differential of that function.

EXAMPLES

1. Find $\int \cos(1 - 3x)dx$.

SOLUTION. Since $1 - 3x$ replaces u in formula (**VIII**), we set $u = 1 - 3x$. Then $du = -3\,dx$ or $dx = -du/3$. Substituting, we get

$$\int \cos(1 - 3x)dx = \int(\cos u)\left(-\frac{du}{3}\right) = -\frac{1}{3}\int \cos u\,du.$$

Therefore

$$\int \cos(1 - 3x)dx = -\frac{1}{3}\sin(1 - 3x) + C.$$

2. Find $\int x \csc(2x^2 - 3)\operatorname{ctn}(2x^2 - 3)\,dx$.

SOLUTION. Using formula (**XII**), we set $u = 2x^2 - 3$. Then

$$du = 4x\,dx, \quad \text{or} \quad x\,dx = du/4.$$

Substituting, we have

$$\int x \csc(2x^2 - 3)\operatorname{ctn}(2x^2 - 3)\,dx = \frac{1}{4}\int \csc u \operatorname{ctn} u\,du$$
$$= -\frac{1}{4}\csc(2x^2 - 3) + C.$$

PROBLEMS

Find the function represented by each of the following integrals.

1. $\int \cos(2 - 3x)dx$. *Ans.* $-(1/3)\sin(2 - 3x) + C$.

2. $\int \sin(3 - 2x)dx$.

3. $\int (2 + \sec^2\theta)d\theta$. *Ans.* $2\theta + \tan\theta + C$.

4. $\int \csc 2\theta \operatorname{ctn} 2\theta\,d\theta$.

5. $\int 4\sin\theta\cos\theta\,d\theta$. *Ans.* $-\cos 2\theta + C$.

6. $\int x\sec^2 x^2 \tan^2 x^2\,dx$.

7. $\int \sqrt{1 + \cos(x/3)}\,dx$. *Ans.* $6\sqrt{2}\sin(x/6) + C$.

8. $\int 3\sin 3x\cos 3x\,dx$.

9. $\int\left(\frac{x}{x^2 + 3} + \sin 2x\right)dx$.
Ans. $(1/2)\log(x^2 + 3) - (1/2)\cos 2x + C$.

10. $\int [e^x + x\cos(2x^2 - 5)]dx$.

11. $\int (x^e - e^x - e^{-x} + e^x\sin e^x)dx$. *Ans.* $\frac{x^{e+1}}{e+1} - e^x + e^{-x} - \cos e^x + C$.

12. $\int [2e^{-2x} + (2x)^e]dx$.

13. $\int dx/(1 - \cos 4\,x)$. *Ans.* $-(1/4)\operatorname{ctn} 2\,x + C$.

14. $\int (e^{-3x} + xe^{x^2} - x^2e^{1-x^3})dx$. $-\frac{e^{-3x}}{3} + \frac{1}{2}e^{x^2} + \frac{e^{1-x^3}}{3} + C$

15. $\int (e^{2x} + 1)^2 e^{2x}\,dx$. *Ans.* $(1/6)(e^{2x} + 1)^3 + C$.

16. $\int (e^{ax} + x^{ae} + 2\,a^{ex})dx$. $\frac{e^{ax}}{a} + \frac{x^{ae+1}}{ae+1} + \frac{2a^{ex}}{e\log a} + C$

17. $\int (x^2 + 2^x + \sec^2 2\,x)dx$.

Ans. $x^3/3 + 2^x/\log 2 + (1/2)\tan 2\,x + C$.

18. $\int e^x(\sin x + \cos x)\,dx/\sqrt{e^x \sin x - 3}$. $2\sqrt{e^x\sin x - 3} + C$

19. $\int \frac{e^{2x}dx}{1 + e^{2x}}$. *Ans.* $(1/2)\log(1 + e^{2x}) + C$.

20. $\int \frac{e^{3x}dx}{(1 - e^{3x})^2}$. $\frac{(1-e^{3x})^{-1}}{3} + C$

21. $\int dx/(e^{4x} + 1)$. *Ans.* $x - (1/4)\log(e^{4x} + 1) + C$.

22. $\int dx/(2 - e^x)$. $-\frac{1}{2}\log(2e^{-x} - 1)$

23. $\int (e^{3x} - 1)dx/(e^{3x} + 1)$. *Ans.* $-x + (2/3)\log(e^{3x} + 1) + C$.

24. $\int dy/(3 - a^y)$.

25. $\int \sqrt{2(1 - \cos 3\,x)}dx$. *Ans.* $-(4/3)\cos(3\,x/2) + C$.

26. $\int \frac{\csc^2\theta - \csc\theta \operatorname{ctn}\theta}{\operatorname{ctn}\theta - \csc\theta}d\theta$. $\log(\csc\theta + \operatorname{ctn}\theta) + C$

27. $\int\left(10^{3x} + 3\,x^{10} + \frac{1}{3^x}\right)dx$. *Ans.* $\frac{10^{3x}}{3\log 10} + \frac{3\,x^{11}}{11} - \frac{3^{-x}}{\log 3} + C$.

28. $\int xe^{3x^2-2}\cos^3 e^{3x^2-2}\,dx$. $\frac{1}{6}\sin(\ ) - \frac{1}{6}\frac{\sin^3(\ )}{3} + C$

CHAPTER XI

METHODS OF INTEGRATION — DEFINITE INTEGRALS

Although we have many formulas of integration from the inverse of known differentiation formulas, there frequently occur integrals which cannot be identified as the differentials of known functions. For that reason we need methods whereby such integrands can be changed into forms more readily identified. A few such methods will now be given.

142. Additional Trigonometric Integrals. The following four formulas are readily derived as shown below.

$$\text{(XIII)} \quad \int \tan u\, du = \int \frac{\sin u\, du}{\cos u}$$
$$= -\log(\cos u) + C, \text{ or } \log(\sec u) + C.$$

$$\text{(XIV)} \quad \int \operatorname{ctn} u\, du = \int \frac{\cos u\, du}{\sin u} = \log(\sin u) + C.$$

$$\text{(XV)} \quad \int \sec u\, du = \int \frac{\sec u(\sec u + \tan u)\, du}{\sec u + \tan u}$$
$$= \log(\sec u + \tan u) + C.$$

$$\text{(XVI)} \quad \int \csc u\, du = \int \frac{\csc u(\csc u - \operatorname{ctn} u)\, du}{\csc u - \operatorname{ctn} u}$$
$$= \log(\csc u - \operatorname{ctn} u) + C.$$

If an integrand is of the form

$$\sin^m u \cos^n u,$$

where m or n is a *positive odd integer*, we can reduce it to powers of either $\sin u$ multiplied by $d(\sin u)$ or $\cos u$ multiplied by $d(\cos u)$.

EXAMPLE

1. Find $\int \cos^2 \theta \sin^3 \theta\, d\theta$.

SOLUTION. Since the exponent of $\sin \theta$ is a positive odd integer, we factor out $\sin \theta\, d\theta$. This leaves $\cos^2 \theta \sin^2 \theta$, which can be changed into terms involving only powers of the $\cos \theta$ as follows.

$$\cos^2 \theta \sin^2 \theta = \cos^2 \theta(1 - \cos^2 \theta) = \cos^2 \theta - \cos^4 \theta.$$

Then let $u = \cos\theta$, whence $du = -\sin\theta\, d\theta$. Therefore, substituting, we have

$$\int \cos^2\theta \sin^3\theta\, d\theta = -\int (u^2 - u^4)\, du = -\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= \frac{3\cos^5\theta - 5\cos^3\theta}{15} + C.$$

If m and n are *both positive even integers*, the double angle formulas can be used to reduce the degree, and this operation must be repeated until an odd power appears. Then treat as explained above when the exponent is a positive odd integer.

EXAMPLE

2. Find $\int \cos^2 2\,\theta\, d\theta$.

SOLUTION. Replace $\cos^2 2\,\theta$ by its equal $(1 + \cos 4\,\theta)/2$. This makes

$$\int \cos^2 2\,\theta\, d\theta = \frac{1}{2}\int (1 + \cos 4\,\theta)\, d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 4\,\theta}{8} + C.$$

In the second term of the new integrand we let $u = 4\,\theta$ and then $d\theta = du/4$. Hence the final result.

For integrands of the forms

$$\tan^n u \quad \text{or} \quad \operatorname{ctn}^n u,$$

where n is an *odd positive integer*, a reduction to sines and cosines is desirable. This makes the integrand come under the type explained in Example 1 above.

However, if n is an *even positive integer*, factor out $\tan^2 u$ or $\operatorname{ctn}^2 u$ and replace by $\sec^2 u - 1$ or $\csc^2 u - 1$ and repeat the operation until all terms are of the form $\tan^m u \sec^2 u\, du$ or $\operatorname{ctn}^m u \csc^2 u\, du$.

EXAMPLE

3. Find $\int \operatorname{ctn}^4\theta\, d\theta$.

SOLUTION. The exponent of $\operatorname{ctn}\theta$ is an even integer and hence we factor out $\operatorname{ctn}^2\theta$ and replace it by $\csc^2\theta - 1$. This gives

$$\int \operatorname{ctn}^4\theta\, d\theta = \int (\operatorname{ctn}^2\theta \csc^2\theta - \operatorname{ctn}^2\theta)\, d\theta.$$

Replacing the second term of this new integrand by $-(\csc^2\theta - 1)$, we have

$$\int \operatorname{ctn}^4\theta\, d\theta = \int (\operatorname{ctn}^2\theta \csc^2\theta - \csc^2\theta + 1)\, d\theta.$$

Now let $u = \text{ctn}\,\theta$, whence $du = -\csc^2\theta\,d\theta$. This makes

$$\int \text{ctn}^4\,\theta\,d\theta = \int (-u^2 + 1)\,du + \int d\theta$$

$$= \theta + \text{ctn}\,\theta - \frac{\text{ctn}^3\,\theta}{3} + C.$$

Many of the more usual trigonometric integrands can be transformed into one of the types discussed in this article.

PROBLEMS

Evaluate each of the following integrals.

1. $\int \tan 2\,\theta\,d\theta.$ *Ans.* $-(1/2)\log\cos 2\,\theta + C.$

2. $\int \text{ctn}\,(\theta/2)d\theta.$ 2 log |~~[illegible]~~ θ| +

3. $\int \sec 3\,\theta\,d\theta.$ *Ans.* $(1/3)\log(\sec 3\,\theta + \tan 3\,\theta) + C.$

4. $\int \csc\,(1 - 2\,\theta)d\theta.$

5. $\int \sin^2(2/3)\,x\,dx.$ *Ans.* $(1/2)[x - (3/4)\sin(4\,x/3)] + C.$

6. $\int \cos^2(1 - 3\,x)dx.$

7. $\int \tan^2(2\,x - 1)dx.$ *Ans.* $(1/2)\tan(2\,x - 1) - x + C.$

8. $\int \text{ctn}^2(3 - 2\,\theta)d\theta.$

9. $\int \sin^3(2\,x + 3)dx.$
Ans. $(1/6)\cos^3(2\,x + 3) - (1/2)\cos(2\,x + 3) + C.$

10. $\int \cos^3(2\,x - 3)dx.$

11. $\int x\tan^3 3\,x^2\,dx.$ *Ans.* $(1/6)[(1/2)\tan^2 3\,x^2 - \log\sec 3\,x^2] + C.$

12. $\int x^2\,\text{ctn}^3(1 - x^3)dx.$

13. $\int \sin^2 2\,\theta\cos^2 2\,\theta\,d\theta.$ *Ans.* $(1/64)(8\,\theta - \sin 8\,\theta) + C.$

14. $\int \sin^2 2\,\theta\cos^3 2\,\theta\,d\theta.$

15. $\int \tan^4\theta\,d\theta.$ *Ans.* $(1/3)\tan^3\theta - \tan\theta + \theta + C.$

16. $\int \sec^4(\theta/2)d\theta.$

17. $\int \cos x\cos 2\,x\,dx.$ *Ans.* $\sin x - (2/3)\sin^3 x + C.$

18. $\int \cos^4 3\,x\,dx.$

19. $\int \tan^3 2\,\theta \sec^4 2\,\theta\, d\theta.$ *Ans.* $(1/24)\,(3 \tan^4 2\,\theta + 2 \tan^6 2\,\theta) + C.$

20. $\int \sin^4 (3 - 2\,x) dx.$

21. $\int \sin^5 (x/3) dx.$
Ans. $-3 \cos (x/3) + 2 \cos^3 (x/3) - (3/5) \cos^5 (x/3) + C.$

22. $\int dx/\sqrt{1 + \cos 2\,x}.$

23. $\int d\theta/\sqrt{1 - \cos 4\,\theta}.$ *Ans.* $(1/2\sqrt{2}) \log (\csc 2\,\theta - \operatorname{ctn} 2\,\theta) + C.$

24. $\int \left(\sqrt{1 + \cos \frac{2\,\theta}{3}}\right)^3 d\theta.$

25. $\int (\sec 2\,\theta - \cos 2\,\theta)^2 d\theta.$
Ans. $(1/2) \tan 2\,\theta - 3\,\theta/2 + (1/8) \sin 4\,\theta + C.$

26. $\int \sqrt{(1 - \cos 3\,\theta)^3}\, d\theta.$

27. $\int \sin^3 \theta \cos^{3/2} \theta\, d\theta.$ *Ans.* $-(2/5) \cos^{5/2} \theta + (2/9) \cos^{9/2} \theta + C.$

28. $\int \cos^3 2\,\theta \sin^{-2/3} 2\,\theta\, d\theta.$

29. $\int \sin^4 (2\,x/3) \cos^3 (2\,x/3) dx.$
Ans. $(3/2)\,[(1/5) \sin^5 (2\,x/3) - (1/7) \sin^7 (2\,x/3)] + C.$

30. $\int \sin^2 2\,x \cos^4 2\,x\, dx.$

31. $\int \sqrt{\sin 2\,x} \cos^5 2\,x\, dx.$
Ans. $(1/3) \sin^{3/2} 2\,x - (2/7) \sin^{7/2} 2\,x + (1/11) \sin^{11/2} 2\,x + C.$

32. $\int \cos^3 (3\,\theta/2) d\theta/\sqrt[3]{\sin (3\,\theta/2)}.$

143. Substitutions for Radicands. If the integrand is rational except for a radical of the form

$$\sqrt[n]{ax + b}, \qquad \text{or} \qquad \sqrt[n]{\frac{ax + b}{cx + d}},$$

the substitution of u for this radical will make the new integrand rational. Integral powers of the radicals mentioned above permit the same substitution.

If the radical is of the form

$$\sqrt[n]{ax^m + b},$$

this same substitution is satisfactory, if only powers of x^m are present, together with $x^{m-1}\,dx$.

EXAMPLES

1. Evaluate $\int x\sqrt{2x-4}\,dx$.

SOLUTION. Let $u = \sqrt{2x-4}$, then $u^2 = 2x - 4$ and $dx = u\,du$. Also $x = (u^2+4)/2$. Substituting, we have

$$\int x\sqrt{2x-4}\,dx = \frac{1}{2}\int (u^2+4)u^2\,du$$

$$= \frac{1}{2}\left(\frac{u^5}{5} + \frac{4u^3}{3}\right) + C.$$

In terms of the original variable this result is

$$\frac{2}{15}(3x^2 - 2x - 8)(2x-4)^{1/2} + C.$$

2. Evaluate $\displaystyle\int \frac{1+x^2}{(3+x)^{1/3}}\,dx.$

SOLUTION. Let $u = (3+x)^{1/3}$, then $x = u^3 - 3$ and $dx = 3u^2\,du$. Also $1 + x^2 = 1 + (u^3-3)^2 = u^6 - 6u^3 + 10$. Therefore

$$\int \frac{1+x^2}{(3+x)^{1/3}}\,dx = \int \frac{u^6 - 6u^3 + 10}{u}\,3u^2\,du = 3\int (u^7 - 6u^4 + 10u)\,du$$

$$= 3\left(\frac{u^8}{8} - \frac{6u^5}{5} + 5u^2\right) + C$$

$$= \frac{3}{40}(5x^2 - 18x + 101)(3+x)^{2/3} + C.$$

3. Evaluate $\displaystyle\int \frac{x^3\,dx}{(8+x^2)^{3/2}}.$

SOLUTION. Let $u = (8+x^2)^{1/2}$, then $u^2 = 8 + x^2$ and $u\,du = x\,dx$. Also $x^2 = u^2 - 8$. Separating $x^3\,dx$ into x^2 and $x\,dx$ and substituting, we get

$$\int \frac{x^3\,dx}{(8+x^2)^{3/2}} = \int \frac{(u^2-8)u\,du}{u^3} = \int (1 - 8u^{-2})\,du$$

$$= u + \frac{8}{u} + C = \frac{16+x^2}{(8+x^2)^{1/2}} + C.$$

PROBLEMS

Find the values of each of the following integrals.

1. $\int x\sqrt{x^2+4}\,dx.$ *Ans.* $(1/3)\sqrt{(x^2+4)^3}+C.$

2. $\int x\,dx/(1+4x^2)^{3/2}.$

3. $\int x^3(1+2x^2)^{1/2}\,dx.$ *Ans.* $(1/30)(3x^2-1)(1+2x^2)^{3/2}+C.$

4. $\int x^3\sqrt{2x^2-1}\,dx.$

5. $\int e^x\sqrt{1-e^x}\,dx.$ *Ans.* $-(2/3)(1-e^x)^{3/2}+C.$

6. $\int x^2(4x+1)^{3/2}\,dx.$

7. $\int x^9(1+2x^5)^{1/2}\,dx.$ *Ans.* $(1/75)(3x^5-1)(1+2x^5)^{3/2}+C.$

8. $\int x\sqrt{x+4}\,dx.$

9. $\int x^2\sqrt{2-3x}\,dx.$
Ans. $-(2/27)(2-3x)^{3/2}[4/3-(4/5)(2-3x)+(1/7)(2-3x)^2]+C.$

10. $\int x\,dx/\sqrt[3]{4-6x^2}.$

11. $\int x\sqrt[3]{3x+1}\,dx.$ *Ans.* $(1/28)(4x-1)(3x+1)^{4/3}+C.$

12. $\int x^5\sqrt{5-2x^3}\,dx.$

13. $\int\left(\dfrac{ax}{\sqrt{5+ax^2}}-a\sin ax\right)dx.$ *Ans.* $\sqrt{5+ax^2}+\cos ax+C.$

14. $\int x^3\,dx/\sqrt{4x^2+7}.$

15. $\int x^2\,dx/\sqrt[3]{(2-x)^2}.$
Ans. $-(3/7)(2-x)^{1/3}(x^2+3x+18)+C.$

16. $\int x^3\,dx/\sqrt{1-x^2}.$

17. $\int x^3\,dx/(2-3x^2)^{3/4}.$ *Ans.* $-(2/45)(2-3x^2)^{1/4}(8+3x^2)+C.$

18. $\int\dfrac{x+2}{\sqrt{8+4x+x^2}}\,dx.$

19. $\int(1+2x^3)\,dx/\sqrt[4]{1-2x-x^4}.$
Ans. $-(2/3)(1-2x-x^4)^{3/4}+C.$

20. $\int x^3$ [redacted] $\sqrt{9\,x^2 - 4}$.

21. $\int \dfrac{\text{[redacted]}}{\sqrt{x^2 - 25}}\,dx.$ *Ans.* $(1/3)(x^2 - 25)^{1/2}(x^2 + 50) + C.$

22. $\int x^2\,dx/(4\,x + 1)^{5/2}$.

144. Integration by Trigonometric Substitutions. If the integrand contains a radical of the form

$$\sqrt{\pm\, a^2 \pm u^2} \tag{1}$$

the substitution of a trigonometric function for u will rationalize this radical. If the radical is

(a) $\sqrt{a^2 - u^2}$, set $u = a \sin \theta$;

(b) $\sqrt{a^2 + u^2}$, set $u = a \tan \theta$;

(c) $\sqrt{u^2 - a^2}$, set $u = a \sec \theta$.

These substitutions make the radicands become some constant multiplied by $1 - \sin^2 \theta$, $1 + \tan^2 \theta$, and $\sec^2 \theta - 1$, respectively. As each of these functions represents the square of a monomial, the expressions are rationalized.

If the radical is of the form

$$\sqrt{a + bx \pm cx^2}, \tag{2}$$

we can factor out the constant c and complete the square so as to get

$$\sqrt{c}\sqrt{\pm\, k \pm (x \pm d)^2}$$

The substitution of u for $(x \pm d)$ changes this radical to form (1). Or directly, if the radical is

(a) $\sqrt{k - (x \pm d)^2}$, set $(x \pm d) = \sqrt{k} \sin \theta$;

(b) $\sqrt{k + (x \pm d)^2}$, set $(x \pm d) = \sqrt{k} \tan \theta$;

(c) $\sqrt{-\,k + (x \pm d)^2}$, set $(x \pm d) = \sqrt{k} \sec \theta$.

EXAMPLES

1. Find $\int \frac{dx}{\sqrt{x^2 + a^2}}$.

Solution. The combination of two positive signs suggests that we set $x = a \tan \theta$. Then $\sqrt{x^2 + a^2}$ becomes $a \sec \theta$ and $dx = a \sec^2 \theta \, d\theta$. Therefore

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a \sec^2 \theta \, d\theta}{a \sec \theta} = \int \sec \theta \, d\theta = \log (\sec \theta + \tan \theta) + C.$$

But $\tan \theta = x/a$ and hence $\sec \theta = \sqrt{x^2 + a^2}/a$. These values in the result above give

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \frac{x + \sqrt{x^2 + a^2}}{a} + C = \log (x + \sqrt{x^2 + a^2}) + K,$$

where $K = C - \log a$.

2. Evaluate $\int \frac{dx}{\sqrt{2x - x^2}}$.

Solution. Completing the square under the radical, we have

$$\sqrt{2x - x^2} = \sqrt{1 - (x - 1)^2}.$$

This suggests that we set $x - 1 = \sin \theta$. Then $\sqrt{2x - x^2} = \cos \theta$ and $dx = \cos \theta \, d\theta$. Substituting, we have

$$\int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{\cos \theta \, d\theta}{\cos \theta} = \int d\theta = \theta + C = \sin^{-1} (x - 1) + C.$$

3. Evaluate $\int \sqrt{2 + 4x - 3x^2}\, dx$.

Solution. Factor out 3 and complete the square, then $\sqrt{2 + 4x - 3x^2}$ becomes $\sqrt{3}\sqrt{2/3 + (4/3)x - x^2}$, which may be written in the form

$$\sqrt{3}\sqrt{(10/9) - (x - 2/3)^2}.$$

Now set $x - 2/3 = (\sqrt{10}/3) \sin \theta$, whence $dx = (\sqrt{10}/3) \cos \theta \, d\theta$ and the new radical becomes $(\sqrt{10}/3) \cos \theta$. Therefore

$$\int \sqrt{2 + 4x - 3x^2}\, dx = \sqrt{3} \int \frac{\sqrt{10}}{3} \cos \theta \cdot \frac{\sqrt{10}}{3} \cos \theta \, d\theta = \frac{10\sqrt{3}}{9} \int \cos^2 \theta \, d\theta$$

$$= \frac{10\sqrt{3}}{9} \int \frac{1}{2} (1 + \cos 2\theta) \, d\theta = \frac{10\sqrt{3}}{9} \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) + C$$

$$= \frac{5\sqrt{3}}{9} \left(\sin^{-1} \frac{3x - 2}{\sqrt{10}} + \frac{3x - 2}{10} \sqrt{6 + 12x - 9x^2} \right) + C.$$

These trigonometric substitutions can be used to change quadratic forms into more suitable expressions even if there is no radical involved, as is illustrated in the following examples.

4. Evaluate $\int \frac{dx}{3x^2 + 5}$.

Solution. Write $3x^2 + 5$ as $3(x^2 + 5/3)$ and set $x = \sqrt{5/3} \tan \theta$. Then $3(x^2 + 5/3) = 5 \sec^2 \theta$ and $dx = \sqrt{5/3} \sec^2 \theta \, d\theta$. Substituting, we get

$$\int \frac{dx}{3x^2 + 5} = \int \frac{\sqrt{5/3} \sec^2 \theta \, d\theta}{5 \sec^2 \theta} = \frac{1}{5}\sqrt{\frac{5}{3}} \int d\theta = \frac{\tan^{-1} x\sqrt{3/5}}{\sqrt{15}} + C.$$

5. Evaluate $\int \frac{dx}{2x^2 - 3}$.

Solution. Write $2x^2 - 3$ as $2(x^2 - 3/2)$ and then let $x = \sqrt{3/2} \sec \theta$. This makes $2x^2 - 3 = 3 \tan^2 \theta$, and $dx = \sqrt{3/2} \sec \theta \tan \theta \, d\theta$. Whence

$$\int \frac{dx}{2x^2 - 3} = \int \frac{\sqrt{3/2} \sec \theta \tan \theta \, d\theta}{3 \tan^2 \theta} = \frac{1}{6}\sqrt{6} \int \csc \theta \, d\theta$$

$$= \frac{1}{6}\sqrt{6} \log (\csc \theta - \operatorname{ctn} \theta) + C$$

$$= \frac{1}{6}\sqrt{6} \log \left(\frac{x\sqrt{2}}{\sqrt{2x^2 - 3}} - \frac{\sqrt{3}}{\sqrt{2x^2 - 3}} \right) + C$$

$$= \frac{1}{6}\sqrt{6} \log \frac{x\sqrt{2} - \sqrt{3}}{\sqrt{2x^2 - 3}} + C.$$

This result can be rationalized in so far as x is concerned by writing it in the form

$$= \frac{1}{12}\sqrt{6} \log \left(\frac{x\sqrt{2} - \sqrt{3}}{\sqrt{2x^2 - 3}} \right)^2 + C$$

$$= \frac{1}{12}\sqrt{6} \log \frac{x\sqrt{2} - \sqrt{3}}{x\sqrt{2} + \sqrt{3}} + C.$$

The first six problems below may be used as formulas of integration if desired.

PROBLEMS

Show that each of the following formulas (Nos. 1–6) is correct.

1. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C,$ when $a > |u|$.

2. $\int \frac{du}{\sqrt{u^2 - a^2}} = \log (u + \sqrt{u^2 - a^2}) + C,$ when $u > |a|$;

$= \log (-u - \sqrt{u^2 - a^2}) + C,$ when $u < -|a|$.

3. $\displaystyle\int \frac{du}{\sqrt{u^2 + a^2}} = \log\,(u + \sqrt{u^2 + a^2}) + C.$

4. $\displaystyle\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C.$

5. $\displaystyle\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \log \frac{u - a}{u + a} + C,$ when $|u| > |a|,$

$\displaystyle = \frac{1}{2a} \log \frac{a - u}{a + u} + C,$ when $|a| > |u|.$

6. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C,$ when $|u| > |a|.$

Evaluate each of the following integrals.

7. $\int dx/\sqrt{2 - x^2}.$ *Ans.* $\sin^{-1}(x/\sqrt{2}) + C.$

8. $\int dx/(x^2 + 4).$

9. $\int dx/\sqrt{4x^2 + 7}.$ *Ans.* $(1/2)\log(2x + \sqrt{4x^2 + 7}) + C.$

10. $\int dx/\sqrt{4x^2 - 5}.$

11. $\int 2\,dx/\sqrt{6 - 4x^2}.$ *Ans.* $\sin^{-1}(2x/\sqrt{6}) + C.$

12. $\int dy/(4y^2 - 1).$

13. $\int 3\,d\theta/\theta\sqrt{2\theta^2 - 3}.$ *Ans.* $\sqrt{3}\sec^{-1}(\theta\sqrt{2/3}) + C.$

14. $\int 3\,d\theta/(\theta^2 - 9).$

15. $\int ds/\sqrt{1 + 2s - 2s^2}.$ *Ans.* $(1/\sqrt{2})\sin^{-1}[(2s - 1)/\sqrt{3}] + C.$

16. $\int dx/\sqrt{2x^2 + 6x + 7}.$

17. $\int dx/\sqrt{4x + 3x^2}.$
Ans. $(1/\sqrt{3})\log(3x + 2 + \sqrt{9x^2 + 12x}) + C.$

18. $\int dx/(3x^2 + 5x + 4).$

19. $\int dx/(3x^2 - 2x).$ *Ans.* $(1/2)\log[(3x - 2)/3x] + C.$

20. $\int dx/\sqrt{5x - 6 - x^2}.$

21. $\int dx/\sqrt{x^2 + 6x + 13}.$ *Ans.* $\log(x + 3 + \sqrt{x^2 + 6x + 13}) + C.$

22. $\int dx/\sqrt{8 - 4x^2 + 4x}$.

23. $\int dx/\sqrt{4x + x^2}$. *Ans.* $\log(x + 2 + \sqrt{x^2 + 4x}) + C$.

24. $\int ds/s\sqrt{3 - s^2}$.

25. $\int d\theta/\theta^2\sqrt{\theta^2 - 4}$. *Ans.* $\sqrt{\theta^2 - 4}/4\theta + C$.

26. $\int dx/x\sqrt{a^2 + x^2}$.

27. $\int x^2\sqrt{1 - x^2}\,dx$. *Ans.* $(1/8)[\sin^{-1} x - x\sqrt{1 - x^2}\,(1 - 2x^2)] + C$.

28. $\int x^2\,dx/(3 - x^2)^{3/2}$.

If the numerator of a fraction with a quadratic denominator involves the form $ax + b$, we suggest the following operations.

EXAMPLES

1. Evaluate $\displaystyle\int \frac{2x - 3}{3x^2 - 2x + 2}\,dx$.

Solution. Factor out the 3 from the denominator and complete the square, so that

$$\int \frac{2x - 3}{3x^2 - 2x + 2}\,dx = \frac{1}{3}\int \frac{(2x - 3)\,dx}{(x^2 - 2x/3 + 1/9) + 2/3 - 1/9}$$
$$= \frac{1}{3}\int \frac{(2x - 3)dx}{(x - 1/3)^2 + 5/9}.$$

Then, as previously, set $x - 1/3 = (\sqrt{5}/3)\tan\theta$. This substitution gives $x = 1/3 + (\sqrt{5}/3)\tan\theta$ and $dx = (\sqrt{5}/3)\sec^2\theta\,d\theta$. Rewriting the integral above, we now have

$$\int \frac{(2x - 3)dx}{3x^2 - 2x + 2} = \frac{1}{3}\int \frac{\frac{2}{3} + \frac{2\sqrt{5}}{3}\tan\theta - 3}{\frac{5}{9}\sec^2\theta}\cdot\frac{\sqrt{5}}{3}\sec^2\theta\,d\theta$$
$$= \frac{1}{3}\int\left(2\tan\theta - \frac{7\sqrt{5}}{5}\right)d\theta$$
$$= \frac{1}{3}\log\sec^2\theta - \frac{7\sqrt{5}}{15}\theta + C$$
$$= \frac{1}{3}\log\frac{9x^2 - 6x + 6}{5} - \frac{7\sqrt{5}}{15}\tan^{-1}\frac{3x - 1}{\sqrt{5}} + C$$
$$= \frac{1}{3}\log(3x^2 - 2x + 2) - \frac{7\sqrt{5}}{15}\tan^{-1}\frac{3x - 1}{\sqrt{5}} + K,$$

where $K = C + \frac{1}{3}\log\frac{3}{5}$.

2. Evaluate $\displaystyle\int \frac{(2x+3)dx}{\sqrt{5+3x-4x^2}}$.

SOLUTION. Factoring out 4 and completing the square in the denominator, we have

$$\int \frac{(2x+3)dx}{\sqrt{5+3x-4x^2}} = \frac{1}{2}\int \frac{(2x+3)dx}{\sqrt{89/64-(x-3/8)^2}}.$$

Now set $x - 3/8 = (\sqrt{89}/8)\sin\theta$, then $x = 3/8 + (\sqrt{89}/8)\sin\theta$ and $dx = (\sqrt{89}/8)\cos\theta\, d\theta$. By substitution,

$$\int \frac{(2x+3)dx}{\sqrt{5+3x-4x^2}} = \frac{1}{2}\int \frac{\frac{3}{4}+\frac{1}{4}\sqrt{89}\sin\theta+3}{\frac{\sqrt{89}}{8}\cos\theta}\cdot\frac{\sqrt{89}}{8}\cos\theta\, d\theta$$

$$= \frac{1}{2}\int\left(\frac{15}{4}+\frac{1}{4}\sqrt{89}\sin\theta\right)d\theta$$

$$= \frac{15}{8}\theta - \frac{\sqrt{89}}{8}\cos\theta + C.$$

Then since $\theta = \sin^{-1}[(8x-3)/\sqrt{89}]$ and $\cos\theta = \sqrt{1-(8x-3)^2/89}$, this becomes in terms of x

$$\int \frac{(2x+3)dx}{\sqrt{5+3x-4x^2}} = \frac{15}{8}\sin^{-1}\frac{8x-3}{\sqrt{89}} - \frac{1}{2}\sqrt{5+3x-4x^2} + C.$$

PROBLEMS

Evaluate each of the following integrals.

1. $\int (x-1)dx/\sqrt{4-x^2}$. *Ans.* $-\sqrt{4-x^2} - \sin^{-1}(x/2) + C$.

2. $\int (3x-1)dx/\sqrt{x^2+4}$.

3. $\int (5x-2)dx/(3x^2-4)$.
Ans. $(5/6)\log(3x^2-4) - (1/2\sqrt{3})\log[(x\sqrt{3}-2)/(x\sqrt{3}+2)] + C$.

4. $\int (2x-3)dx/(5x^2+7)$.

5. $\int (3-x)dx/\sqrt{3x-4x^2}$.
Ans. $(3/16)(7\sin^{-1}[(8x-3)/3] + (4/3)\sqrt{3x-4x^2}) + C$.

6 $\int (3-2t)dt/(4t^2-2t+3)$.

7. $\int (x + 1)dx/\sqrt{8 + 4x + x^2}$.

Ans. $\sqrt{x^2 + 4x + 8} - \log(x + 2 + \sqrt{x^2 + 4x + 8}) + C$.

8. $\int (2s + 3)ds/\sqrt{15 + 10s - 5s^2}$.

9. $\int (3x - 2)dx/\sqrt{5x^2 - 10x - 15}$.

Ans. $(1/\sqrt{5}) \log (x - 1 + \sqrt{x^2 - 2x - 3}) + (3/\sqrt{5})\sqrt{x^2 - 2x - 3} + C$.

10. $\int (4\theta - 3)d\theta/\sqrt{1 + 3\theta - 3\theta^2}$.

11. $\int (5x - 2)dx/\sqrt{8 - 4x^2 + 4x}$.

Ans. $(1/4) \sin^{-1}[(2x - 1)/3] - (5/2)\sqrt{2 - x^2 + x} + C$.

12. $\int (4\theta + 3)d\theta/\sqrt{-2 - 3\theta + 4\theta^2}$.

145. Integration by Parts. The formula for the differential of the product $u \cdot v$ gives a very useful integration formula. Since

$$d(u \cdot v) = u \cdot dv + v \cdot du,$$

we get, by integrating each term,

$$\int d(u \cdot v) \equiv u \cdot v = \int u \cdot dv + \int v \cdot du.$$

This is used most frequently when written in the form

$$\textbf{(XVII)} \qquad \int u\, dv = uv - \int v\, du.$$

This formula allows us to replace a difficult integral, represented in the formula by $\int u\, dv$, by $uv - \int v\, du$, where $\int v\, du$ may be more readily evaluated. Obviously, the usefulness of the formula depends upon the proper choice of u and dv. Since we need both du and v, *the function chosen to represent dv should be one which is readily integrated.* Also, it is advisable to choose as the function u that part of the integrand which is simplified by differentiation.

Formula (**XVII**) is specially useful when the integrand is composed of the product of two types of functions; that is, the product of an algebraic and a trigonometric, a logarithmic and an exponential, an inverse trigonometric and an algebraic function, or other such combinations. Such integrals lend themselves to ***integration by parts,*** as the application of this formula is called.

Some general suggestions can be made which will usually help the student to satisfactory choices for u and dv.

(a) *An integrand composed of the product of a positive integral power of some variable x and either a trigonometric or an exponential function of the variable will generally be simplified by letting u represent the power of x and dv the remainder of the expression, provided this remainder can be readily integrated.* This choice is made because the differential of a power of a variable is of lower degree than the original, while the differentials of trigonometric or exponential functions are no simpler than the original functions. It may be necessary to apply the formula several times before the form $\int v\,du$ becomes a simple integral.

(b) *An integrand which involves a logarithmic or inverse trigonometric function is simplified by letting u represent such a function.* This choice removes such functions since the differential of $\log u$ or of any inverse trigonometric function is an algebraic function.

(c) *If the integrand is the product of an exponential and a trigonometric function, such as e^{ax} and either $\sin bx$ or $\cos bx$, the formula must be applied twice and the final relation solved for the desired integral.* In such repeated applications, set u equal to either of the functions for the first application, and in the second set u equal to the function appearing for the derivative of u in the first application.

EXAMPLES

1. Integrate $\int xe^{2x}\,dx$.

SOLUTION. The integrand is the product of a power of x and the exponential e^{2x}. Therefore we set $u = x$ and $dv = e^{2x}\,dx$. Whence $du = dx$ and $v = (1/2)e^{2x}$. Substituting in formula (**XVII**), we get

$$\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$$

2. Evaluate $\int x\sin^{-1}x\,dx$.

SOLUTION. Since $\sin^{-1}x$ is present, we let $u = \sin^{-1}x$ and $dv = x\,dx$. Then $du = dx/\sqrt{1-x^2}$ and $v = x^2/2$. These give

$$\int x\sin^{-1}x\,dx = \frac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int\frac{x^2dx}{\sqrt{1-x^2}}.$$

The second integral is similar to those we have solved by means of a trigonometric substitution. Hence we let $x = \sin\theta$, so that $dx = \cos\theta\, d\theta$ and $\sqrt{1 - x^2} = \cos\theta$. Therefore the second integral $\int \frac{x^2dx}{\sqrt{1 - x^2}}$ becomes

$$\int \frac{\sin^2\theta\cos\theta\, d\theta}{\cos\theta} = \int \sin^2\theta\, d\theta = \frac{1}{2}\int (1 - \cos 2\,\theta)\, d\theta$$

$$= \frac{\theta}{2} - \frac{\sin 2\,\theta}{4} + C.$$

But $\theta = \sin^{-1} x$ and $\sin 2\,\theta = 2\sin\theta\cos\theta = 2\,x\sqrt{1 - x^2}$. Therefore

$$\int \frac{x^2dx}{\sqrt{1 - x^2}} = \frac{1}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1 - x^2} + C,$$

which makes

$$\int x \sin^{-1} x\, dx = \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{1}{4}x\sqrt{1 - x^2} + C.$$

3. Find $\int e^{3x} \cos 2\,x\, dx$.

Solution. This comes under (*c*) above and we let $u = e^{3x}$ or $\cos 2\,x$ as differentiation does not simplify either function. If $u = e^{3x}$, we may write $dv = \cos 2\,x\, dx$, whence $du = 3\,e^{3x} dx$ and $v = (1/2)\sin 2\,x$. These give

$$\int e^{3x} \cos 2\,x\, dx = \frac{1}{2}e^{3x}\sin 2\,x - \frac{3}{2}\int e^{3x}\sin 2\,x\, dx. \tag{1}$$

This second integral is no simpler than the original and so we repeat the operation by letting $u = e^{3x}$ and $dv = \sin 2\,x\, dx$. Then $du = 3\,e^{3x}\, dx$ and $v = -(1/2)\cos 2\,x$. These in the second integral of (1) give

$$\begin{aligned}\int e^{3x} \cos 2\,x\, dx &= \frac{1}{2}e^{3x}\sin 2\,x - \frac{3}{2}\left[-\frac{1}{2}e^{3x}\cos 2\,x + \frac{3}{2}\int e^{3x}\cos 2\,x\, dx\right]\\ &= \frac{1}{2}e^{3x}\sin 2\,x + \frac{3}{4}e^{3x}\cos 2\,x - \frac{9}{4}\int e^{3x}\cos 2\,x\, dx.\end{aligned}$$

Transposing the last integral to the left and dividing by its resulting coefficient, 13/4, we have

$$\int e^{3x}\cos 2\,x\, dx = \frac{e^{3x}}{13}(2\sin 2\,x + 3\cos 2\,x) + C.$$

4. Evaluate $\int \sqrt{a^2 + x^2}\, dx$.

Solution. This is not a mixture of two types of functions; however, it is readily integrated by parts.

Thus, let $u = \sqrt{a^2 + x^2}$ and $dv = dx$. Whence $du = x\,dx/\sqrt{a^2 + x^2}$ and $v = x$. Therefore

$$\int \sqrt{a^2 + x^2}\, dx = x\sqrt{a^2 + x^2} - \int \frac{x^2 dx}{\sqrt{a^2 + x^2}}$$

$$= x\sqrt{a^2 + x^2} - \int \frac{x^2 + a^2 - a^2}{\sqrt{a^2 + x^2}}\, dx$$

$$= x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2}{\sqrt{a^2 + x^2}}\, dx + a^2 \int \frac{dx}{\sqrt{a^2 + x^2}}$$

$$= x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2}\, dx + a^2 \log (x + \sqrt{a^2 + x^2}).$$

Transposing and dividing by 2, we get

$$\int \sqrt{a^2 + x^2}\, dx = \frac{1}{2} x\sqrt{a^2 + x^2} + \frac{1}{2} a^2 \log (x + \sqrt{a^2 + x^2}) + C.$$

5. Evaluate $\int \sec^3 \theta\, d\theta$.

SOLUTION. This is another common integral which may be integrated by parts.

Set $u = \sec \theta$ and $dv = \sec^2 \theta\, d\theta$. Then $du = \sec \theta \tan \theta\, d\theta$ and $v = \tan \theta$. Therefore

$$\int \sec^3 \theta\, d\theta = \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta\, d\theta$$

$$= \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1)\, d\theta$$

$$= \sec \theta \tan \theta - \int \sec^3 \theta\, d\theta + \int \sec \theta\, d\theta$$

$$= \sec \theta \tan \theta - \int \sec^3 \theta\, d\theta + \log (\sec \theta + \tan \theta).$$

Whence

$$\int \sec^3 \theta\, d\theta = \frac{1}{2} [\sec \theta \tan \theta + \log (\sec \theta + \tan \theta)] + C.$$

In the last three examples the constant of integration has not been added until all other operations have been completed. This is done merely to prevent the necessity of changing the form of the constant several times.

PROBLEMS

Evaluate each of the following integrals.

1. $\int \tan^{-1} x\, dx$. *Ans.* $x \tan^{-1} x - (1/2) \log (1 + x^2) + C$.

2. $\int \sin^{-1} 2\, x\, dx$.

3. $\int x \sin x\, dx$. *Ans.* $- x \cos x + \sin x + C$.

4. $\int x \cos^{-1} x\, dx.$

5. $\int (x + 2)e^{x+3}dx.$ *Ans.* $e^{x+3}(x + 1) + C.$

6. $\int \log s\, ds.$

7. $\int \log t\, dt/t^3.$ *Ans.* $-(1/2\,t^2)\log t - 1/(4\,t^2) + C.$

8. $\int u^2 e^{3u} du.$

9. $\int x^2 \log 2\,x\, dx.$ *Ans.* $(1/9)\,x^3\,(3 \log 2\,x - 1) + C.$

10. $\int \theta^2 \sin 2\,\theta\, d\theta.$

11. $\int x^2 \sin^{-1} 2\,x\, dx.$

Ans. $(1/3)x^3 \sin^{-1} 2\,x + (1/36)(1 + 2\,x^2)\sqrt{1 - 4\,x^2} + C.$

12. $\int e^{2t} \sin t\, dt.$

13. $\int e^{3x} \cos 3\,x\, dx.$ *Ans.* $(1/6)e^{3x}\,(\sin 3\,x + \cos 3\,x) + C.$

14. $\int \sqrt{4\,\theta^2 - 12}\, d\theta.$

15. $\int \sqrt{4\,x^2 + 1}\, dx.$

Ans. $(1/2)[x\sqrt{4\,x^2 + 1} + (1/2) \log (2\,x + \sqrt{4\,x^2 + 1})] + C.$

16. $\int \theta^2 \cos 3\,\theta\, d\theta.$

17. $\int x \sec^{-1} 3\,x\, dx.$ *Ans.* $(1/2)x^2 \sec^{-1} 3\,x - (1/18)\sqrt{9\,x^2 - 1} + C.$

18. $\int \sec^3 3\,\theta\, d\theta.$

19. $\int d\theta/(\sin^3 2\,\theta).$

Ans. $(1/4)[-\csc 2\,\theta \operatorname{ctn} 2\,\theta + \log (\csc 2\,\theta - \operatorname{ctn} 2\,\theta)] + C.$

20. $\int x^2\,(\log x)^2\, dx.$

21. $\int x^3\sqrt{a^2 - x^2}\, dx.$ *Ans.* $-[5\,x^3(a^2 - x^2)^{3/2} + 2\,(a^2 - x^2)^{5/2}]/15 + C.$

22. $\int xe^{-x}\, dx/(1 - x)^2.$

146. Integration of Rational Fractions. An improper rational fraction should be reduced to a mixed number before integration. The proper fraction resulting from division may be separated into partial fractions whose denominators are the factors of the original denominator.

We shall explain the method of partial fractions for three different types by means of examples.

(*a*) *When the factors of the denominator are linear and not repeated.*

EXAMPLE

1. Evaluate $\int \frac{5x^2 - 7x + 8}{x^3 + 3x^2 - 4x} dx.$

SOLUTION 1. This is a proper fraction and the factors of the denominator are x, $x + 4$, and $x - 1$. Therefore, we assume that

$$\frac{5x^2 - 7x + 8}{x^3 + 3x^2 - 4x} \equiv \frac{A}{x} + \frac{B}{x+4} + \frac{C}{x-1}, \tag{1}$$

where A, B, and C are unknown constants. Clearing (1) of fractions and collecting the coefficients of the powers of x on the right, we have

$$5x^2 - 7x + 8 \equiv (A + B + C)x^2 + (3A - B + 4C)x - 4A. \tag{2}$$

For two polynomials to be identically equal, it is necessary that the coefficients of corresponding powers of the variable be equal. Equating coefficients, we get

$$\begin{cases} A + B + C = 5, \\ 3A - B + 4C = -7, \\ -4A = 8. \end{cases} \tag{3}$$

Solving equations (3) simultaneously, we find that $A = -2$, $B = 29/5$, $C = 6/5$. Hence

$$\int \frac{5x^2 - 7x + 8}{x^3 + 3x^2 - 4x} dx = \int \left(-\frac{2}{x} + \frac{29}{5(x+4)} + \frac{6}{5(x-1)} \right) dx$$

$$= -2\int \frac{dx}{x} + \frac{29}{5}\int \frac{dx}{x+4} + \frac{6}{5}\int \frac{dx}{x-1}$$

$$= -2 \log x + \frac{29}{5} \log (x + 4) + \frac{6}{5} \log (x - 1) + C.$$

SOLUTION 2. In clearing (1) of the preceding solution of fractions we get

$$5x^2 - 7x + 8 \equiv A(x + 4)(x - 1) + Bx(x - 1) + Cx(x + 4). \tag{4}$$

Since this relation must be true for all values of x, we proceed to use special values so as to get equations involving A, B, and C which are more quickly solved than those of (3) above. Thus we notice that $x = 0$ removes the terms that involve B and C, and so the resulting equation, $8 = -4A$, is solved at once, and $A = -2$; similarly $x = 1$ removes A and B terms and

we get $6 = 5C$, or $C = 6/5$; also $x = -4$ gives $116 = 20B$, or $B = 29/5$. The remainder of the solution is the same as that given above.

(*b*) *When the factors of the denominator are linear and some are repeated.*

EXAMPLE

2. Evaluate $\int \frac{4x^2 - 7x + 10}{(x+2)(3x-2)^2}\,dx$.

SOLUTIONS. Assume

$$\frac{4x^2 - 7x + 10}{(x+2)(3x-2)^2} \equiv \frac{A}{x+2} + \frac{Bx + C}{(3x-2)^2}.$$

But $Bx + C = B'(3x - 2) + C'$, where $B' = B/3$, and $C' = (2B + 3C)/3$. Hence we have

$$\frac{4x^2 - 7x + 10}{(x+2)(3x-2)^2} \equiv \frac{A}{x+2} + \frac{B'}{3x-2} + \frac{C'}{(3x-2)^2}. \tag{5}$$

Here we have assumed that the original fraction is separated into two proper fractions having the factors of the original denominator as their denominators. Then we show that the last fraction can be separated into two fractions and so *all integral powers of each factor of the given denominator up to and including the power to which it appears in the denominator may be used as denominators of partial fractions.* We also point out that *the numerators remain constants.* Clearing (5) of fractions and using either method outlined in Example 1, we find that $A = 5/8$, $B' = -13/24$, $C' = 8/3$. Therefore

$$\int \frac{4x^2 - 7x + 10}{(x+2)(3x-2)^2}\,dx = \frac{5}{8}\log(x+2) - \frac{13}{72}\log(3x-2) - \frac{8}{9(3x-2)} + C.$$

(*c*) *When some factors of the denominator are of the second degree.*

EXAMPLE

3. Evaluate $\int \frac{x^3 - 3x^2 + 5x - 12}{(x-1)^2(x^2 + 3x - 2)}\,dx$.

SOLUTIONS. Since $x - 1$ is a repeated linear factor, we shall assume partial fractions of the types $A/(x-1)$ and $B/(x-1)^2$. Then, since the other factor of the denominator is a quadratic, we assume a numerator $Cx + D$ which is the next lower degree. Thus

$$\frac{x^3 - 3x^2 + 5x - 12}{(x-1)^2(x^2 + 3x - 2)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2 + 3x - 2},$$

and, as shown in the other examples, the solution is completed.

A rational trigonometric function of an angle x may be transformed into a rational algebraic fraction in u by the substitution

(1) $$\tan \frac{x}{2} = u.$$

For then $\sin \frac{x}{2} = \frac{u}{\sqrt{1+u^2}}, \qquad \cos \frac{x}{2} = \frac{1}{\sqrt{1+u^2}},$

whence

(2) $$\sin x = \frac{2u}{1+u^2}, \qquad \cos x = \frac{1-u^2}{1+u^2}.$$

Also

(3) $$x = 2\tan^{-1} u, \qquad dx = \frac{2\,du}{1+u^2}.$$

EXAMPLE

4. Evaluate $\int \sec x\, dx$.

SOLUTION. Since $\sec x = \frac{1}{\cos x} = \frac{1+u^2}{1-u^2}$ and $dx = \frac{2\,du}{1+u^2}$, the integral becomes $2\int du/(1-u^2)$, which is readily integrated.

PROBLEMS

1. Prove the following formula by the use of partial fractions.

$$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\log\frac{u-a}{u+a}, \quad |u|>|a|, \quad \text{or} \quad \frac{1}{2a}\log\frac{a-u}{a+u}, \quad |u|<|a|.$$

Evaluate each of the following integrals.

2. $\int dx/(x^2 - 6x + 5)$.

3. $\int (2x-1)dx/(x^2-1)$. *Ans.* $\log (x-1)^{1/2}(x+1)^{3/2} + C$.

4. $\int (3x+2)dx/(x^2+x)$.

5. $\int dx/(6x - 9x^2 + 15)$. *Ans.* $(1/3)\log (x+1)^{1/8}(5-3x)^{-1/8} + C$.

6. $\int (2x+1)dx/(x^3 - x)$.

7. $\int dx/(1-\sin x)$. *Ans.* $2/[1-\tan (x/2)] + C$.

8. $\int dx/(1+\cos x)$.

9. $\int 2\,x\,dx/[(x+2)(x^2-1)]$.

Ans. $\log (x+1)(x-1)^{1/3}(x+2)^{-4/3}+C$.

10. $\int dx/[x(x^2+2\,x+1)]$.

11. $\int (2\,x+1)dx/[x(x^2+1)]$. Ans. $2\tan^{-1} x + \log x(x^2+1)^{-1/2}+C$.

12. $\int (x+3)dx/[(x+1)(x^2+1)]$.

13. $\int dx/(4+2\cos x)$. Ans. $(1/\sqrt{3})\tan^{-1}\left(\dfrac{\tan (x/2)}{\sqrt{3}}\right)+C$.

14. $\int d\theta/(3-2\sin\theta)$.

15. $\int (x+3)dx/[x(x^2-6\,x+5)]$. Ans. $\log x^{3/5}(x-5)^{2/5}(x-1)^{-1}+C$.

16. $\int (3\,x^2-x+8)dx/[(x+1)(x^2+5)]$.

17. $\int x^4\,dx/(x^2-1)$. Ans. $x^3/3+x+(1/2)\log[(x-1)/(x+1)]+C$.

18. $\int (1+2\,x-3\,x^3)dx/(2\,x^3+2\,x)$.

19. $\int (x^4+1)dx/(x^3+x)$. Ans. $(1/2)\,x^2+\log[x/(x^2+1)]+C$.

20. $\int \csc\theta\,d\theta$.

21. $\int (\operatorname{ctn}\theta\,d\theta)/(1-\cos\theta)$.

Ans. $-(1/4)\operatorname{ctn}^2(\theta/2)-(1/2)\log\tan(\theta/2)+C$.

22. $\int (\tan\theta\,d\theta)/(1-\cos\theta)$.

23. $\int (x^4+11\,x-6)dx/(x^3+8)$.

Ans. $(1/2)x^2+\log[\sqrt{x^2-2\,x+4}/(x+2)]+C$.

24. $\int (x^4+2\,x^3+2\,x^2+2\,x+1)dx/(x^3+x^2+x+1)$.

25. $\int (\sin^2\theta\,d\theta)/(1-\sin\theta)$. Ans. $\cos\theta-\theta+2/[1-\tan(\theta/2)]+C$.

147. Use of Integral Tables. The student should realize by now that integration is largely an individual process and readily accomplished or not according to one's choice of a suitable method.

We have tried to offer enough suggestions as to methods to enable one to evaluate most of the more usual integrals. However, a *Table of Integrals* is often necessary or at least very convenient. Such tables do not have all possible integrals in them, hence the transformations previously proposed may be needed to bring the given integral into the form which may be found in the tables.

Often this may be done by merely replacing some function of the variable present by a new variable. Then again it may be necessary to use the formula for integration by parts or the process outlined for rational fractions before the given integral is changed into one which may be found in the tables at the end of this book.

148. The Definite Integral. Usually, as in the case of a moving body, we are not interested in the whole distance the body has traveled but in the distance it has traveled during a definite interval of time. This is also generally true in all problems of integration. Thus, consider the function y derived from the relation

$$(1) \qquad dy = dF(x) = f(x)dx.$$

We have

$$(2) \qquad y = \int f(x)dx = F(x) + C.$$

For $x = a$ this function has the value $F(a) + C$, and for $x = b$ its value is $F(b) + C$. Hence, if x has changed continuously from the value a to the value b, the total change in the function is

$$(3) \qquad [F(b) + C] - [F(a) + C] = F(b) - F(a).$$

This difference of the values of the indefinite integral for $x = a$ and $x = b$ is called the ***definite integral of $f(x)dx$*** between the ***limits*** a and b. It is written

$$(4) \qquad \int_a^b f(x)dx = F(x)\Big]_a^b \equiv F(b) - F(a).$$

Notice that since the constant of integration disappears in taking the difference of the two values of the indefinite integral, it is not written when the integral has limits.

EXAMPLES

1. If $dy = 2x^2 dx$, find the change in y from $x = 2$ to $x = 4$.

SOLUTION. Using integral notation, we have

$$y = \int 2x^2 dx.$$

Then the change in y is represented by

$$\int_2^4 2x^2 dx = \frac{2x^3}{3}\Big]_2^4 = \frac{128}{3} - \frac{16}{3} = 37\tfrac{1}{3} \text{ units}.$$

2. The velocity of a falling body is gt ft./sec. How far will the body fall during the third and fourth seconds? ($g = 32.2$ ft./sec.²)

SOLUTION. Since $v = ds/dt = gt$, we have $ds = gt\,dt$. Therefore

$$s = \int gt\,dt.$$

Then the change in s, or the distance fallen, is

$$s_2 - s_1 = \int_2^4 gt\,dt = \frac{1}{2}gt^2\Big]_2^4 = 6\,g = 193.2 \text{ ft.}$$

If a change of variable, due to a substitution to facilitate integration, is used, the limits must be changed to correspond to the new variable. However, if such new limits are not of convenient form for evaluating the integrated function, this function may be changed back to the original variable and the original limits used. In such cases, the limits should appear on the integral sign during the process of integration but written so as to refer to the original variable, as illustrated in Example 4 below.

3. Evaluate $\int_{-a/2}^{a/2} \frac{du}{\sqrt{a^2 - u^2}}$.

SOLUTION. In this example we set $u = a \sin\theta$, then $du = a \cos\theta\,d\theta$ and $\sqrt{a^2 - u^2} = a\cos\theta$. When $u = -a/2$, the given substitution makes $\sin\theta = -1/2$, or $\theta = -\pi/6$, and when $u = a/2$, $\sin\theta = 1/2$, so $\theta = \pi/6$. Now the example can be written

$$\int_{-a/2}^{a/2} \frac{du}{\sqrt{a^2 - u^2}} = \int_{-\pi/6}^{\pi/6} \frac{a\cos\theta\,d\theta}{a\cos\theta}$$

$$= \int_{-\pi/6}^{\pi/6} d\theta = \theta\Big]_{-\pi/6}^{\pi/6} = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}.$$

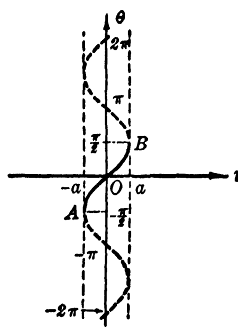


FIG. 147

In general an integral has one and only one value at each limit, but here the value $\theta = \sin^{-1}(u/a)$ has an infinite number of values at each limit. We must be careful in selecting the values to be used. A part of the graph of $\theta = \sin^{-1}(u/a)$ is shown in Fig. 147. In the integral u is supposed to vary continuously from $-a/2$ to $a/2$. This condition is satisfied if we choose values for θ or $\sin^{-1}(u/a)$ from the part AB of the curve. That is, for negative values of u, we take $\sin^{-1}(u/a)$ between $-\pi/2$ and 0, and for positive values of u, we use $\sin^{-1}(u/a)$ as evaluated between 0 and $\pi/2$.

4. Evaluate $\int_2^5 x\sqrt{x-2}\,dx$.

SOLUTION. To integrate this example set $u = \sqrt{x-2}$. Then $u^2 = x - 2$, $2\,u\,du = dx$, and $x = u^2 + 2$. Also for $x = 2, u = 0$, and for $x = 5, u = \sqrt{3}$. Hence

$$\int_2^5 x\sqrt{x-2}\,dx = 2\int_0^{\sqrt{3}} (u^2+2)u^2\,du = 2\left(\frac{u^5}{5} + \frac{2\,u^3}{3}\right)\Big]_0^{\sqrt{3}} = \frac{38\sqrt{3}}{5}.$$

The substitution used in such integrals, here $u = \sqrt{x-2}$, must be one which changes in one direction throughout the interval of integration. Thus, as x increases continuously from 2 to 5, the new function u increases throughout the interval from 0 to $\sqrt{3}$. Care must be taken in substitution so that this is true.

The original limits may be used if the following notation is carried out:

$$\begin{aligned}\int_2^5 x\sqrt{x-2}\,dx &= 2\int_{x=2}^{x=5} (u^4 + 2\,u^2)du \\ &= 2\left(\frac{u^5}{5} + \frac{2\,u^3}{3}\right)\Big]_{x=2}^{x=5} \\ &= 2\left[\frac{1}{5}(x-2)^{5/2} + \frac{2}{3}(x-2)^{3/2}\right]_2^5 \\ &= \frac{38\sqrt{3}}{5}.\end{aligned}$$

5. Find the value of $\int_{-a}^{a} \frac{dx}{x^2+a^2}$.

SOLUTION. Let $x = a\tan\theta$, then $dx = a\sec^2\theta\,d\theta$, and the denominator, $x^2 + a^2 = a^2\sec^2\theta$. Also for $x = -a$, $\tan\theta = -1$, or $\theta = -\pi/4$, and for $x = a$, $\tan\theta = 1$, so $\theta = \pi/4$. Therefore

$$\begin{aligned}\int_{-a}^{a}\frac{dx}{x^2+a^2} = \int_{-\pi/4}^{\pi/4}\frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta} &= \frac{1}{a}\int_{-\pi/4}^{\pi/4} d\theta = \frac{1}{a}\theta\Big]_{-\pi/4}^{\pi/4} \\ &= \frac{\pi}{4\,a} - \left(-\frac{\pi}{4\,a}\right) = \frac{\pi}{2\,a}.\end{aligned}$$

Again from $d\theta = dx/(x^2 + a^2)$ we get

$$\frac{1}{a}\theta\Big]_{x=-a}^{x=a} = \int_{-a}^{a}\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\Big]_{-a}^{a}$$

which has an infinite number of values at each limit. Consider the graph of

$$\theta = \tan^{-1}\frac{x}{a}$$

of Fig. 148. We must be careful to select proper values for the integral. In the integral x varies continuously from $-a$ through 0 to $+a$. This requires that we move along a single branch from $x = -a$ to $x = a$. It is most convenient to use the branch which passes through the origin; then for a negative value of x we must choose θ or $\tan^{-1}(x/a)$ between $-\pi/2$ and 0, and for a positive value of x the value of $\tan^{-1}(x/a)$ must be between 0 and $\pi/2$.

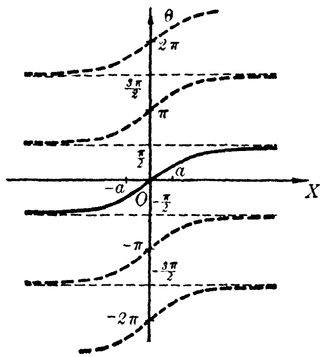


FIG. 148

6. Evaluate $\displaystyle\int_b^c \frac{dx}{x\sqrt{x^2 - a^2}}$.

SOLUTION. This form of integral suggests the substitution $x = a \sec\theta$. This makes $\sqrt{x^2 - a^2} = a \tan\theta$, $dx = a \sec\theta \tan\theta \, d\theta$; therefore

$$\int_b^c \frac{dx}{x\sqrt{x^2 - a^2}} = \int_{x=b}^{x=c} \frac{a \sec\theta \tan\theta \, d\theta}{a \sec\theta \;\; a \tan\theta} = \frac{1}{a}\int_{x=b}^{x=c} d\theta = \frac{1}{a}\theta\Big]_{x=b}^{x=c}$$

$$= \frac{1}{a}\sec^{-1}\frac{x}{a}\Big]_b^c = \frac{1}{a}\left[\sec^{-1}\frac{c}{a} - \sec^{-1}\frac{b}{a}\right].$$

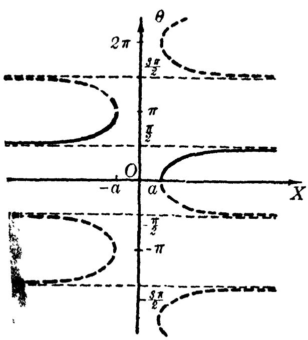


FIG. 19

Here again

$$\frac{1}{a}\theta\Big]_{x=b}^{x=c} = \frac{1}{a}\sec^{-1}\frac{x}{a}\Big]_b^c$$

has an infinite number of values at each limit. Fig. 149 shows the graph of $y = \theta = \sec^{-1}(x/a)$. Since x varies continuously from b to c, θ must vary continuously and to do this $\sec^{-1}(x/a)$ can be taken between 0 and $\pi/2$ for positive values b and c. Then for negative vaiues, say $-b$ $-c$, the values of $\sec^{-1}(x/a)$ should be taken between and π.

PROBLEMS

Evaluate of the following definite integrals.

1. $\int_1^8$ a 'x. *Ans.* 3.

2. $\int_c$ $+ x^2)$.

3. $'(9 - x^2)^{1/2}$. *Ans.* $3 - (3/2)\sqrt{3}$.

4. $\int_0^a x^3\,dx/(2\,a - x)$.

5. $\int_{-1/2}^{3/2} d\theta/(4\,\theta^2 + 3)$. *Ans.* $\pi/4\sqrt{3}$.

6. $\int_0^{\sqrt[3]{19}} x^2\,dx/\sqrt[3]{x^3 + 8}$.

7. $\int_2^5 dx/(2\,x^2 - 5\,x + 3)$. *Ans.* $\log\,(7/4)$.

8. $\int_2^4 x^3\sqrt{x^2 - 4}\,dx$.

9. $\int_2^5 s\sqrt{1 + s}\,ds$. *Ans.* $(4\sqrt{3}/5)(13\sqrt{2} - 2)$.

10. $\int_0^5 dx/\sqrt{9 - x}$.

11. $\int_{-1}^0 d\theta/\sqrt{4 - \theta^2}$ *Ans.* $\pi/6$.

12. $\int_{-1/3}^{1/3} dx/\sqrt{4 - 9\,x^2}$.

13. $\int_0^{\pi/2} \theta\sin\theta\,d\theta$. *Ans.* 1.

14. $\int_{-1}^2 dx/\sqrt{4 - 2\,x^2}$.

15. $\int_{-3/2}^{-1/2} dx/(4\,x^2 + 8\,x + 7)$. *Ans.* $\pi/(6\sqrt{3})$

16. $\int_1^2 \sqrt{2\,x - x^2}\,dx$.

17. $\int_{-1/2}^{1/2} dx/\sqrt{7 + 4\,x - 4\,x^2}$. *Ans.* $\pi/8$.

18. $\int_0^{1.5} (9 - x^2)^{3/2}dx$.

19. $\int_0^{\pi/4} \cos^3 2\,x\,dx$. *ns.* 1 3.

20. $\int_{-1/2}^0 x^2\sqrt{1 - x^2}\,dx$.

21. $\int_{-1/2}^1 \sin^{-1} t\,dt$. $12 - \sqrt{3}/2$.

22. $\int_{-1}^0 dx/(1 + \sqrt{1 + x})$.

23. $\int_{1/2\sqrt{2}}^{\sqrt{3}/4} dx/(x^2\sqrt{1 - 4\,x^2})$. $\sqrt{3} - 1)$.

24. $\int_0^\pi \theta\sin^2\theta\,d\theta$.

149. Interchange of the Limits of the Definite Integral. Since the symbol

$$\int_a^b f(x)\,dx \quad \text{represents} \quad F(b) - F(a),$$

where

$$\int f(x)\,dx = F(x) + C,$$

and

$$\int_b^a f(x)\,dx \quad \text{represents} \quad F(a) - F(b),$$

it is evident that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

150. Subintervals of Integration. Since

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{and} \qquad \int_b^c f(x)\,dx = F(c) - F(b),$$

it is evident that

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

A SUPPLEMENTARY GROUP OF INTEGRALS

1. $\int (y/4 - 3)^2\,dy.$ *Ans.* $(4/3)(y/4 - 3)^3 + C.$

2. $\int (4 - x^2)\,dx/x^4.$

3. $\int \dfrac{x^3\,dx}{2x+1}.$
   Ans. $(1/48)[8x^3 - 6x^2 + 6x - 3\log(2x+1)] + C.$

4. $\int 3x^3\,dx/(1 - x).$

5. $\int \left(\dfrac{5x}{x^2 - 3} - \sin 2x\right) dx.$
   Ans. $(1/2)[5\log(x^2 - 3) + \cos 2x] + C.$

6. $\int y(a + by^2)^{-1}\,dy.$

7. $\int x\,dx/\sqrt{a^2 - x^2}.$ *Ans.* $-\sqrt{a^2 - x^2} + C.$

8. $\int (x/\sqrt{5 - ax^2} + \sin ax)\,dx.$

9. $\int 5\,bx\,dx/(8\,a - 6\,bx^2)$. *Ans.* $-(5/12)\log(8\,a - 6\,bx^2) + C$.

10. $\int x^2\,dx/\sqrt{a^3 + x^3}$.

11. $\int \operatorname{ctn}^2(1 - 2\,\theta)\csc^2(1 - 2\,\theta)\,d\theta$. *Ans.* $(1/6)\operatorname{ctn}^3(1 - 2\,\theta) + C$.

12. $\int \tan^4\theta\sec^4\theta\,d\theta$.

13. $\int \sec^6(2\,\theta)\tan^2(2\,\theta)\,d\theta$.
Ans. $(1/210)[35\tan^3(2\,\theta) + 42\tan^5(2\,\theta) + 15\tan^7(2\,\theta)] + C$.

14. $\int (1 + 2\,x^3)\,dx/\sqrt[4]{1 - 2\,x - x^4}$.

15. $\int dx/(1 + e^{3x})$. *Ans.* $x - (1/3)\log(1 + e^{3x}) + C$.

16. $\int e^{2x}\,dx/(1 - e^x)$.

17. $\int \cos a\theta\,d\theta/(1 + \sin a\theta)$. *Ans.* $(1/a)\log(1 + \sin a\theta) + C$.

18. $\int \sin ax\cos ax\,dx/(1 - \cos 2\,ax)$.

19. $\int (x + 2)^2\,dx/x^3$. *Ans.* $\log x - 4/x - 2/x^2 + C$.

20. $\int \sqrt{b^2\cos^2\theta + a^2\sin^2\theta}\,\sin\theta\cos\theta\,d\theta$.

21. $\int x\,dx/\sqrt{x - 1}$. *Ans.* $(2/3)(x + 2)\sqrt{x - 1} + C$

22. $\int \sqrt{x + 1}\,dx/x$.

23. $\int x\sqrt{2\,x + 1}\,dx$. *Ans.* $(1/15)(6\,x^2 + x - 1)\sqrt{2\,x + 1}$ C

24. $\int \sqrt{\theta^2 + 1}\,d\theta/\theta$.

25. $\int x^3\,dx/\sqrt{1 - x^2}$. *Ans.* $-(1/3)(x^2 + 2)\sqrt{1 - x^2} + C$

26. $\int (a^{2/3} - x^{2/3})^2\,dx/x^{1/3}$.

27. $\int x^2\,dx/(x^2 + 1)^2$. *Ans.* $(1/2)\tan^{-1}x - \dfrac{x}{2(x^2 + 1)}$

28. $\int (3\,x - 2)\,dx/\sqrt{9 - x^2}$.

29. $\int \sin^5\theta\cos^2\theta\,d\theta$.
Ans. $(1/105)\cos^3\theta\,(35 - 42\cos^2\theta + 15\cos^4\theta) + C$.

30. $\int \sin^{1/3}(x/3)\cos^3(x/3)\,dx$.

31. $\int \log x\,dx/x$. *Ans.* $(1/2)(\log x)^2 + C$.

32. $\int \log x \, dx/x^2$.

33. $\int \cos^4 (x/5) \, dx$.
Ans. $(1/16) [20 \sin (x/5) \cos^3 (x/5) + 15 \sin (2x/5) + 6x] + C$.

34. $\int \cos^2 (1 - 2x) \sin^2 (1 - 2x) \, dx$.

35. $\int e^x \, dx/(1 - e^x)^2$. *Ans.* $1/(1 - e^x) + C$.

36. $\int e^{2x} \cos 3x \, dx$.

37. $\int dx/\sqrt{3x^2 + 4x - 10}$.
Ans. $(1/\sqrt{3}) \log (3x + 2 + \sqrt{9x^2 + 12x - 30}) + C$.

38. $\int x \log (x^2 + 1) \, dx$.

39. $\int (a^2 - x^2)^{3/2} \, dx$.
Ans. $(1/8)[(a^2 - x^2)^{1/2}(5a^2x - 2x^3) + 3a^4 \sin^{-1} (x/a)] + C$.

40. $\int x \sec^{-1} (x/2) \, dx$.

41. $\int dx/\sqrt{2x^2 - 3}$. *Ans.* $(1/\sqrt{2}) \log (x + \sqrt{x^2 - 3/2}) + C$.

42. $\int dx/\sqrt{1 + 4x - x^2}$.

43. $\int dx/(x^2 + 6x - 13)$. *Ans.* $(1/2\sqrt{22}) \log \dfrac{x + 3 - \sqrt{22}}{x + 3 + \sqrt{22}} + C$.

44. $\int 6 \, dx/(x^2 - 4)$.

45. $\int x \, dx/[(x + 1)(x^2 + 1)]$.
Ans. $\log (x + 1)^{-1/2}(x^2 + 1)^{1/4} + (1/2) \tan^{-1}x + C$.

46. $\int dx/(7 + 4x - 4x^2)$.

47. $\int x^5 \, dx/\sqrt{2x^3 + 1}$. *Ans.* $(1/9)(2x^3 + 1)^{1/2}(x^3 - 1) + C$.

48. $\int dx/\sqrt{2x^2 + 3x}$.

49. $\int x \, dx/\sqrt[4]{2x + 6}$. *Ans.* $(1/7)(2x + 6)^{3/4}(2x - 8) + C$.

50. $\int dx/\sqrt{x^2 - 3x + 1}$.

51. $\int dx/\sqrt{7 + 5x - 2x^2}$. *Ans.* $(1/\sqrt{2}) \sin^{-1} [(4x - 5)/9] + C$.

52. $\int \cos^2 \theta \, d\theta/(1 - \cos \theta)$.

53. $\int x \cos^{-1} (2x) \, dx$.
Ans. $(x^2/2) \cos^{-1} (2x) + (1/16) \sin^{-1} (2x) - (x/8)\sqrt{1 - 4x^2} + C$.

54. $\int \sqrt{x^2 - 12}\, dx.$

55. $\int y\sqrt{a^{2/3} - y^{2/3}}\, dy.$

Ans. $-(1/35)(8\,a^{4/3} + 12\,a^{2/3}y^{2/3} + 15\,y^{4/3})(a^{2/3} - y^{2/3})^{3/2} + C.$

56. $\int (y^4 - 5\,y^3 + 6\,y - 4)\, dy/(y^3 - 4\,y^2 - 5\,y).$

57. $\int y^2\, dy/[(y - 1)(y^2 + 1)].$

Ans. $(1/2) \log (y - 1) + (1/4) \log (y^2 + 1) + (1/2) \tan^{-1} y + C.$

58. $\int \sqrt{y^2 - 1}\, dy/y^3.$

59. $\int dx/\sqrt{(x^2 - 9)^3}.$ *Ans.* $-x/(9\sqrt{x^2 - 9}) + C.$

60. $\int (x^4 + 2\,x^3 + 2\,x^2 + 2\,x + 2)\, dx/[(x + 1)(x^2 + 1)].$

61. $\int 5\,x\, dx/\sqrt{1 - x^4}.$ *Ans.* $(5/2) \sin^{-1} x^2 + C.$

62. $\int \cos^3 (2\,x - 1)\, dx.$

63. $\int \sqrt{y^2 - 1}\, dy/y.$ *Ans.* $(y^2 - 1)^{1/2} - \tan^{-1} (y^2 - 1)^{1/2} + C.$

64. $\int (x + 6)\, dx/\sqrt{1 + 2\,x - 3\,x^2}.$

65. $\int \tan^3 (2 - 3\,x)\, dx.$

Ans. $-(1/6) \sec^2 (2 - 3\,x) - (1/3) \log [\cos (2 - 3\,x)] + C.$

66. $\int \tan^3 (2\,x/3) \sec^4 (2\,x/3)\, dx.$

67. $\int \tan \theta\, d\theta/(1 + \cos \theta).$ *Ans.* $-\log [1 - \tan^2 (\theta/2)] + C.$

68. $\int \operatorname{ctn} \theta\, d\theta/(1 + \sin \theta).$

69. $\int \sin mx \cos nx\, dx, \quad m \neq n.$

Use $\sin (A + B) + \sin (A - B) = 2 \sin A \cos B.$

Ans. $-\frac{1}{2}\left[\frac{\cos (m + n)x}{m + n} + \frac{\cos (m - n)x}{m - n}\right] + C.$

70. $\int \cos mx \cos nx\, dx, \ m \neq n.$

71. $\int \sin mx \sin nx\, dx, \ m \neq n.$

Ans. $-\frac{1}{2}\left[\frac{\sin (m + n)x}{m + n} - \frac{\sin (m - n)x}{m - n}\right] + C.$

72. $\int a^2b^2\, d\theta/(a^2 \sin^2 \theta + b^2 \cos^2 \theta).$

73. $\int \sqrt{\sin \theta}\, d\theta/\cos^{5/2} \theta.$ *Ans.* $(2/3) \tan^{3/2} \theta + C.$

74. $\int \sqrt{(1 + \sqrt{x})/x}\, dx.$

75. $\int x \tan^{-1} 2\,x\, dx.$

Ans. $(1/2)\, x^2 \tan^{-1} x - x/4 + (1/8) \tan^{-1} 2\,x + C.$

76. $\int x \sin^{-1} 3x\, dx.$

77. $\int \sin^4 (3 - 2x)\, dx.$

Ans. $[4 \sin^3 (3 - 2x) \cos (3 - 2x) - 6(3 - 2x) + 3 \sin (6 - 4x)]/32 + C.$

78. $\int \cos^3 3x\, dx/(1 - \cos 3x).$

79. $\int_0^{\pi/2} \sin \theta \cos \theta\, d\theta.$ *Ans.* $1/2.$

80. $\int_0^1 [4x/(x^2 + 4) - x/2]\, dx.$

81. $\int_0^{\pi/4} \cos^5 (2x)\, dx.$ *Ans.* $4/15.$

82. $\int_2^5 dx/(2x^2 - x - 3).$

83. $\int_0^a \sqrt{a^{1/3} + x^{1/3}}\, dx.$ *Ans.* $4(11\sqrt{2} - 4)a^{7/6}/35.$

84. $\int_0^{\pi/4} \sec^2 \theta \sqrt{\tan \theta + 3}\, d\theta.$

85. $\int_0^{\pi} \sin \theta\, (1 - \theta + \sin \theta)\, d\theta.$ *Ans.* $2 - \pi/2.$

86. $\int_1^2 (e^{2x} - 1)\, dx/(e^{2x} + 1).$

87. $\int_0^2 x\sqrt{4 - x^2}\, dx.$ *Ans.* $8/3.$

88. $\int_0^{\pi/2} \sin^3 (\theta/2)\, d\theta.$

89. $\int_0^{\pi} \sin^6 (\theta/2)\, d\theta.$ *Ans.* $5\pi/16.$

90. $\int_1^e \sqrt{1 + \log x}\, dx/x.$

91. $\int_{-\pi/4}^{\pi/4} \cos \theta \sqrt{1 + \sin^2 \theta}\, d\theta.$ *Ans.* $\sqrt{3}/2 + (1/2) \log \dfrac{\sqrt{3} + 1}{\sqrt{3} - 1}.$

92. $\int_{-2}^2 x^2\sqrt{4 - x^2}\, dx.$

93. $\int_0^{\log 4} dx/\sqrt{e^{-2x} + 2e^{-x}}.$ *Ans.* $3 - \sqrt{3}.$

94. $\int_{-a}^a a\, dx/[x\sqrt{a^2 + x^2}].$

95. If $dy/dx = \sec^2 x$, what is $\int_{x=0}^{x=\pi/4} dy$? *Ans.* 1.

96. $\int_0^r y\, dx$ for $x = \sqrt{2ry - y^2}.$

97. $\int_0^r \sqrt{1 + (dx/dy)^2}\, dy$ for $x = \sqrt{2ry - y^2}.$ *Ans.* $\pi r/2.$

98. $\int_0^r y\sqrt{1 + (dx/dy)^2}\, dy$ for $x = \sqrt{2\,ry - y^2}$.

99. $\int_0^a \sqrt{1 + (dy/dx)^2}\, dx$ for $x^{2/3} + y^{2/3} = a^{2/3}$. *Ans.* $3\,a/2$.

100. $\int_0^a y\sqrt{1 + (dy/dx)^2}\, dx$ for $y = b + \sqrt{a^2 - x^2}$.

101. $\int_0^5 xy\sqrt{1 + (dy/dx)^2}\, dx$ for $9\,x^2 + 25\,y^2 = 225$. *Ans.* $245/8$.

102. $\int_0^a y\sqrt{1 + (dy/dx)^2}\, dx$ for $8\,a^2y^2 = x^2(a^2 - x^2)$.

103. $\int_a^x f'(y)\, dy$ if $f'(y) = (d/dy)f(y)$. *Ans.* $f(x) - f(a)$.

104. $\int_a^x f''(y)(x - y)\, dy$. (Use parts.)

105. $\int_a^x [f'''(y)(x - y)^2/2!]\, dy$.

Ans. $f(x) - f(a) - f'(a)(x - a) - f''(a)(x - a)^2/2!$.

106. $\int_a^x f^{n+1}\,(y)(x - y)^n\, dy/n!$. Get several terms.

CHAPTER XII

APPLICATIONS AND INTERPRETATIONS OF INTEGRALS

151. Curves with Given Properties. If the equation of a curve is given as $F(x, y) = 0$ or $\phi(r, \theta) = 0$, we can find the differentials of these functions. The derived equation will usually involve x, y, dx, and dy or r, θ, dr, and $d\theta$, and is called a ***differential equation.***

On the other hand, if a differential equation is given which we can reduce to the form

$$f_1(y)\, dy = f_2(x)\, dx,$$

or

$$\phi_1(r)\, dr = \phi_2(\theta)\, d\theta,$$

in which the variables are said to be ***separated,*** we may be able to integrate each member of such a relation and thereby obtain the *equation of the curve except for the constant of integration.* Some additional information is necessary to find the constant so as to know definitely the equation of the curve. If the constant of integration cannot be found definitely, we can consider it as a parameter and then the equation represents a ***family*** or ***system of curves,*** one curve for each definite value assigned the constant. Such a system of curves is called a ***one-parameter system*** as the constant of integration plays the role of the parameter.

EXAMPLES

1. If the slope of a curve at any point is the square of the reciprocal of the abscissa of the point and if the curve passes through (2, 4), find its equation.

SOLUTION. We are given that

$$\frac{dy}{dx} = \frac{1}{x^2}, \tag{1}$$

which written in differential notation becomes

$$dy = \frac{dx}{x^2}. \tag{2}$$

Integrating each member of this equation, we have

$$(3) \qquad y = -\frac{1}{x} + C,$$

where C represents both constants of integration, one of which might have been written on each side of the equation.

The equation (3) represents a family of curves such that the slope of each satisfies the condition (1). To find the member of this family that passes through (2, 4), we impose the condition that (2, 4) satisfy (3) and thereby find that $C = 9/2$. Therefore the desired equation is

$$y = -\frac{1}{x} + \frac{9}{2}, \quad \text{or} \quad 2\,xy - 9\,x + 2 = 0,$$

which is a *rectangular hyperbola.*

2. Find the equation of the curve which is perpendicular to the line joining any point of itself to the point (3, 4), if it passes through the origin.

SOLUTION. Let P_1 represent the point (3, 4) and $P(x, y)$ any point of the curve. The slope of the curve at P is the negative reciprocal of that of the line P_1P. Therefore

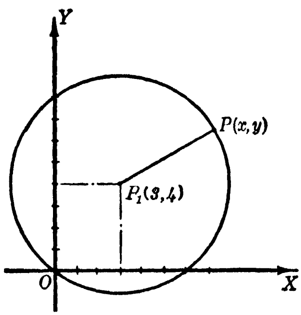


FIG. 150

$$\frac{dy}{dx} = -\frac{x-3}{y-4},$$

whence, separating the variables, we find

$$(y-4)dy = (3-x)dx.$$

Integration gives the system of curves

$$\frac{y^2}{2} - 4\,y = 3\,x - \frac{x^2}{2} + C.$$

Since the curve we want passes through the origin, we substitute $x = 0$, $y = 0$ in this equation and find that $C = 0$. Clearing and transposing, we obtain

$$x^2 + y^2 - 6\,x - 8\,y = 0,$$

a *circle* with (3, 4) as its center.

3. Find the equation of the curve through the point $(a, \pi/4)$, from which is derived the relation $dr/d\theta = (a^2/r)\cos 2\,\theta$.

SOLUTION. Separating the variables r and θ, the given relation can be written as follows,

$$r\,dr = a^2 \cos 2\,\theta\,d\theta.$$

Integrating, we get

$$\frac{r^2}{2} = \frac{a^2}{2}\sin 2\,\theta + C.$$

Then substitution of $(a, \pi/4)$ in this gives

$$\frac{a^2}{2} = \frac{a^2}{2} \sin \frac{\pi}{2} + C,$$

whence $C = 0$. Therefore the desired equation is

$$r^2 = a^2 \sin 2\,\theta.$$

PROBLEMS

1. The slope of a curve at any point is x^2 and it passes through (1, 5). Find its equation. *Ans.* $3\,y = x^3 + 14$.

Find the equation of the curve which satisfies the following conditions. (Nos. 2–6.)

2. Slope -5 and passes through the point $(-2, 8)$.

3. Slope $2\,x$, through the point (1, 3). *Ans.* $y = x^2 + 2$.

4. Slope $2\,x/(1 + x^2)$, through the point (0, 0).

5. Slope $2\,y$, through the point (0, 5). *Ans.* $2\,x = \log\,(y/5)$.

6. Slope y^2, through the point (1, 5).

7. The rate of change of the slope of a curve with respect to x is $2 - 6\,x$. Find its equation if it passes through $(1, -1)$ and $(-1, 5)$. *Ans.* $y = 1 - 2\,x + x^2 - x^3$.

8. The slope of a curve is proportional to the ordinate at any point. If it passes through (1, 3) with the slope 3/2, what is its equation?

9. The slope of a curve at any point is three units less than the square of twice the abscissa of the point. If it passes through (2, 7), what is its equation? *Ans.* $3\,y = 4\,x^3 - 9\,x + 7$.

10. The rate of change of the slope of a curve with respect to x is $3\,x$. Find its equation if it passes through the points (2, 3) and $(-2, 11)$.

11. Find the equation of the curve for which $d^2y/dx^2 = 5$, and $m = 3$ at the point $(2, -4)$. *Ans.* $2\,y = 5\,x^2 - 14\,x$.

12. The same as Problem 11 for $d^2y/dx^2 = 4 - 2\,x$, if y has a minimum value at the point (1, 2).

13. The same as Problem 11 for $dm/dx = \cos x$, and $m = 3$ at the point $(\pi/2, 2)$. *Ans.* $y = 2\,x - \cos x + 2 - \pi$.

14. The same as Problem 11 for $dm/dx = 2 \sin x$, and $m = 4$ at the point $(\pi/2, 2)$.

15. The same as Problem 11 for $y'' = 4 - x^2$, and $m = 1$ at the point $(2, -10/3)$. *Ans.* $12\,y = 24\,x^2 - x^4 - 52\,x - 16$.

16. Find the polar equation of the curves which make an angle with each radius vector equal to that which locates the radius vector.

17. Find the equation in polar coordinates of the curves which make the same angle with the radius vector to each point. *Ans.* $r = ce^{k\theta}$.

18. Find the equation of the curve through $(0, c)$ which has its tangent of constant length c. The length of the tangent to a curve is $y\sqrt{1 + (dx/dy)^2}$.

19. Find the equation of the curve through $(1, 5)$ which intersects the curves $y = (1/2) \log (1 - x^2) + c$ at right angles.

Ans. $2y = 2 \log x - x^2 + 11$.

152. Straight Line Motion. If the distance a body moves along a straight line is given as a function of the time, we have found the velocity and acceleration to be ds/dt and d^2s/dt^2, respectively. We are now able to reverse the problem. That is, given the velocity or acceleration of a body as a function of the time, we may find the distance the body has moved and its direction of motion along its path at any time t. The constants of integration which appear during the inverse operations are determined by initial conditions or additional information.

EXAMPLES

1. If the acceleration of a body is $6 - 3t$, find the distance the body will move after $t = 0$ while the velocity is increasing; given the initial condition $v = 0$ when $t = 0$.

SOLUTION. The velocity increases during the interval in which the acceleration is positive. For this example this interval begins at $t = 0$ and continues until $t = 2$. Hence the distance moved during the interval from $t = 0$ to $t = 2$ is desired. The acceleration

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = 6 - 3t.$$

Then

$$dv = (6 - 3t)dt,$$

$$v = 6t - \frac{3t^2}{2} + C.$$

Since $v = 0$ when $t = 0$, the constant of integration C is zero.

Before evaluating s over the interval from $t = 0$ to $t = 2$, we must see if the motion is in the same direction during the interval. Since the direction of motion may change when v is zero, we see from

$$v = 6t - \frac{3t^2}{2} = \frac{3}{2}(4 - t)t = 0$$

that the direction of motion can change only at $t = 0$ or $t = 4$. Hence it

does not change between $t = 0$ and $t = 2$. Now, writing s as a definite integral, we have

$$s = \int_0^2 \left(6\,t - \frac{3\,t^2}{2}\right) dt$$

$$= 3\,t^2 - \frac{t^3}{2}\Big]_0^2 = 12 - 4 = 8 \text{ units.}$$

2. If $v = 2 - 3\,t + t^2$, find the distance traveled by a particle during the interval from $t = 1$ to $t = 5$.

SOLUTION. The velocity is zero at $t = 1$ and $t = 2$. Since it is negative from $t = 1$ to $t = 2$, the motion of the particle during that time is opposite to the direction in which positive s is measured. To find the total distance traveled, we must compute s for the intervals from $t = 1$ to $t = 2$ and from $t = 2$ to $t = 5$, and add the two results numerically. That is, let

$$s_1 = \int_1^2 (2 - 3\,t + t^2)dt \quad \text{and} \quad s_2 = \int_2^5 (2 - 3\,t + t^2)dt.$$

Whence

$$s_1 = \left(2\,t - \frac{3\,t^2}{2} + \frac{t^3}{3}\right)\Big]_1^2 = \left(4 - 6 + \frac{8}{3}\right) - \left(2 - \frac{3}{2} + \frac{1}{3}\right) = -\frac{1}{6},$$

and

$$s_2 = \left(2\,t - \frac{3\,t^2}{2} + \frac{t^3}{3}\right)\Big]_2^5 = \left(10 - \frac{75}{2} + \frac{125}{3}\right) - \frac{2}{3} = 13\tfrac{1}{2}.$$

Therefore the total distance traveled is

$$13\frac{1}{2} + \frac{1}{6} = 13\frac{2}{3} \text{ units.}$$

PROBLEMS

1. The velocity of a particle along a straight line is $2\,t - 4$. How far does it move from $t = 1$ to $t = 4$? *Ans.* 5 units.

2. The same as Problem 1, with $v = t^2 - 6\,t + 5$, $t = 0$, $t = 3$.

3. The same as Problem 1, with $v = t^2 - 6\,t + 5$, $t = 3$, $t = 7$. *Ans.* 16 units.

4. The same as Problem 1, with $v = t^2 + 2\,t - 3$, $t = 0$, $t = 3$.

5. The same as Problem 1, with $v = 10 - 7\,t + t^2$, $t = 0$, $t = 6$. *Ans.* 15 units.

6. The same as Problem 1, with $v = -2\,t^2 + 11\,t - 12$, $t = 3$, $t = 7$.

7. The same as Problem 1, with $v = 12(t^3 - 4\,t^2 + 3\,t)$, $t = 0$, $t = 4$. *Ans.* 96 units.

8. The velocity of a body moving along a straight line is $3t - t^2$. How far does it move while the velocity is positive?

9. The velocity of a body along a straight line is $10 - 7t + t^2$. How far does it move in the direction opposite to that in which the distance is measured?

Ans. $4\frac{1}{2}$ units.

10. The same as Problem 9 for $v = t^2 - 3t + 2$.

11. The acceleration of a body along a straight line is $t^2 - 4t$. How much does its velocity change during the interval in which the acceleration is negative?

Ans. $10\frac{2}{3}$ units.

12. The velocity of a body along a straight line is $t - 2$. How far apart are its positions at $t = 1$ and $t = 4$? How far has it moved?

13. The same as Problem 12, with $v = t^2 - 6t + 8$, $t = 1$, $t = 3$.

Ans. 2/3 units; 2 units.

14. The same as Problem 12, with $v = 2t^2 - 7t + 5$, $t = 0$, $t = 2$.

15. The velocity of a body along a straight line is $18 - 3t$. How far will it move from $t = 2$ until it stops?

Ans. 24 units.

16. If $v = e^3 - e^{2t}$, how far does the body move after $t = 0$ while $v > 0$?

153. Projectiles. Let the point of projection of a projectile be taken as the origin of coordinates, the x axis horizontal, and the y axis directed upward. Suppose the initial velocity is v_0 and the angle of elevation is α.

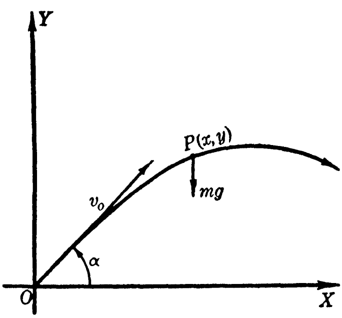


FIG. 151

If we assume that the force of gravity alone acts on the projectile, there is no horizontal force and hence the horizontal velocity dx/dt is constant and the horizontal acceleration d^2x/dt^2 is zero.

The vertical force acting downward being that of gravity, we have

$$\frac{d^2y}{dt^2} = \frac{dy'}{dt} = -g, \tag{1}$$

$$dy' = -g\,dt.$$

Integrating (1), the vertical component of the velocity is found to be

$$y' = \frac{dy}{dt} = -gt + C. \tag{2}$$

The initial conditions are

$$(3) \qquad \begin{cases} t = 0, \quad x = 0, \quad y = 0, \\ \dfrac{dx}{dt} = v_0 \cos \alpha, \quad \dfrac{dy}{dt} = v_0 \sin \alpha, \end{cases}$$

whence C of (2) is $v_0 \sin \alpha$.

Then, since the horizontal component is constant, these components of the velocity are

$$(4) \qquad \frac{dx}{dt} = v_0 \cos \alpha, \qquad \frac{dy}{dt} = -gt + v_0 \sin \alpha.$$

Integrating relations (4), we have

$$\begin{cases} x = v_0 t \cos \alpha + C_1, \\ y = v_0 t \sin \alpha - \dfrac{1}{2} g t^2 + C_2. \end{cases}$$

The initial conditions $x = 0$, $y = 0$, $t = 0$ make $C_1 = C_2 = 0$. Hence the path of the projectile is given parametrically by

$$(5) \qquad \boldsymbol{x = v_0 t \cos \alpha, \qquad y = v_0 t \sin \alpha - \frac{1}{2} g t^2.}$$

As the velocity of the projectile in its path is

$$v = \sqrt{v_x^2 + v_y^2},$$

we have, from (4),

$$v = \sqrt{v_0^2 \cos^2 \alpha + v_0^2 \sin^2 \alpha - 2\, v_0 g t \sin \alpha + g^2 t^2},$$

or

$$(6) \qquad \boldsymbol{v = \sqrt{v_0^2 - 2\, v_0 g t \sin \alpha + g^2 t^2}.}$$

In the following problems the student should derive the paths needed by integrations and not by substitution in formulas (5).

PROBLEMS

1. Find the rectangular equation of the path of a projectile of 200 ft./sec. initial velocity at an elevation of $\pi/6$. *Ans.* $y = x/\sqrt{3} - gx^2/60000$.

2. Find the time of flight on a horizontal plane of a projectile of initial velocity v_0 and elevation α.

3. Show that the range of a projectile on a horizontal plane is $(v_0^2 \sin 2\alpha)/g$. What elevation makes the range a maximum? *Ans.* $\pi/4$.

4. To what maximum height will a projectile rise?

5. When is the velocity of a projectile least? *Ans.* $t = (v_0 \sin \alpha)/g$.

6. Find the ranges on an inclined plane of inclination θ, the projectile being fired directly up and down the plane.

7. Find the elevation for a maximum range for Problem 6.
Ans. $\alpha = \theta/2 + \pi/4$.

8. A plane flying 100 mi./hr. at a height of 1/2 mile will drop a bomb on a target. How long before passing over the target should the bomb be released?

9. If it is 60 ft. from the position of a pitcher's hand when he releases a ball to the batter, and if the ball leaves his hand horizontally and 5 ft. above the ground, how slowly may it leave his hand to pass by the batter 2 ft. above the ground? *Ans.* $80\sqrt{3}$ ft./sec.

10. A projectile of initial velocity v_0 ft./sec. is to hit a target k ft. away on the same plane as the point of projection. What angle or angles of elevation are necessary?

11. Assuming $\alpha = 60°$, $v_0 = 75$ ft./sec., $g = 32$ ft./sec.^{2}, find v_x, v_y, and v of a projectile if $x = 20$ ft. *Ans.* 37.5 ft./sec.; 47.89 ft./sec.; 60.82 ft./sec.

12. Solve Problem 11 if $y = 30$ ft. instead of $x = 20$ ft.

13. Find the envelope of the paths of projectiles fired from a given point with the same muzzle velocity but with different angles of elevation.
Ans. $2y = v_0^2/g - gx^2/v_0^2$.

154. The Law of Natural Growth. If bacteria are allowed to grow naturally with sufficient food supply, the increase per second in the number in a cubic unit of culture is proportional to the number in that cubic unit. The ***law of natural growth*** may be derived from this type of rate of change. Thus, if the rate of change of a quantity y with respect to an independent variable t is proportional to y itself, we have

$$\frac{dy}{dt} = ky.$$

Therefore

$$\frac{dy}{y} = k\,dt.$$

Integrating, we find

$$\log y = kt + \log C,$$

where the constant of integration is taken as $\log C$ so as to simplify the next operations. Then

$$\log y - \log C = \log \frac{y}{C} = kt,$$

and hence

$$\frac{y}{C} = e^{kt},$$

or

$$(1) \qquad y = Ce^{kt}.$$

This is the ***law of natural growth.***

Some instances in which this law applies are: The rate of decrease of air pressure as we leave the surface of the earth is proportional to the pressure at each height. The rate of increase of the number of bacteria in a culture at any time is proportional to the number present at that time. The rate of change of the difference of the temperatures of a body and a cooling flow of air or liquid is proportional to that difference. This law applies even to compound interest if the interest is assumed to be continuously compounded. Such is the case for an investment which is continually increasing, or decreasing at a given rate.

The student should derive the law by integration to fit each problem given below.

EXAMPLES

1. When light enters a medium, its rate of absorption with respect to the depth of penetration t is proportional to the amount of light incident on a unit area at the depth. Find the law connecting L and t if the incident light is L_0 and the emerging light after passing through a thickness t_1 is L_1.

Solution. We are given that

$$\frac{dL}{dt} = -kL.$$

Whence

$$\frac{dL}{L} = -k\,dt,$$

and integrating, we have

$$\log L = -kt + \log C.$$

Therefore

$$L = Ce^{-kt}.$$

But when $t = 0$, $L = L_0$, then $C = L_0$. Also when $t = t_1$, $L = L_1$; hence $L_1 = L_0e^{-kt_1}$, or

$$e^{-k} = \left(\frac{L_1}{L_0}\right)^{1/t_1},$$

whence

$$L = L_0\left(\frac{L_1}{L_0}\right)^{t/t_1}.$$

2. Suppose an amount of money A at a rate of interest r is changing continuously by instantaneous compounding. Find the law connecting A and the time t it is at interest.

SOLUTION. We have, to begin with, the compound interest law

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt},$$

where A_0 is the original amount invested, r the rate of interest, t the number of years, and n the number of compoundings per year. If the student is not familiar with this law he can readily derive the special cases where $n = 1, 2, \cdots$ and thereby see how the law is derived for any n.

We can rewrite the law in the form

$$A = A_0\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt}.$$

To let the compounding occur instantaneously is to let $n \to \infty$, and we have

$$A = A_0e^{rt},$$

since $\lim\limits_{n\to\infty} (1 + r/n)^{n/r} = e$. This is the natural growth law because

$$\frac{dA}{dt} = r(A_0e^{rt}) = rA.$$

The example as stated is equivalent to an investment of value A which is continually increasing at a rate r per year.

PROBLEMS

1. Sugar decomposes at a rate proportional to the amount present. If 50 lbs. becomes 16.4 lbs. in 4 hrs., when will 0.05 lb. remain?

Ans. In 24.79 hrs.

2. If the rate of growth of a quantity is proportional to itself and if it doubles in 10 yrs., when will it treble?

3. The rate of change of v with respect to t is proportional to v. If $v = 10.5$ units at $t = 0$, and 26.25 units at $t = 5$, find its value at $t = 6$.

Ans. 31.53 units.

4. A substance s decomposes at a rate proportional to itself. If $s = 3.24$ units when $t = 0$, and 1.62 units when $t = 3$, find s for $t = 5$.

5. A sheet hung in the wind loses its moisture at a rate proportional to the moisture remaining. If one half is lost in 1 hr. what part remains after 5 hrs.? Check by a progression. *Ans.* 1/32 of the original amount.

6. A rotating wheel is slowing down in such a way that the angular acceleration is proportional to the angular velocity. If $\omega = 75$ r. p. s. at the beginning of the slowing down and in 2 min. it is 3 r. p. s., at what time is $\omega = 30$ r. p. s.?

7. A cube of dry ice evaporates at a rate proportional to the surface of the block. How long before a 6 in. cube reduces to a 1 in. cube, if an 8 in. cube reduces to 6 inches in 1 hr.? *Ans.* 2.5 hrs.

8. If the population changes at a rate proportional to itself, what population will a city of 40,000 have in 20 years, if 20 years ago it was 25,000?

9. The rate of decrease of a quantity is proportional to the quantity. If 1/4 disappears in 2 hrs., when will 1/10 remain? *Ans.* At $t = 16$ hrs.

10. The rate of decrease of a quantity is proportional to the quantity. If 1/5 disappears in 2 hrs., when will 1/9 remain?

11. If 100 lbs. of one substance is transformed into another at a rate proportional to the amount untransformed, and after 3 hrs., 51.2 lbs. is not transformed, derive a formula for the amount transformed at any time.

Ans. $s = 100(4/5)^t$.

12. The funds of an institution over \$300,000 are appreciating in value at the rate of 4% per annum. If the original amount is \$800,000, when will it be doubled?

13. The same as Problem 8; if the population increased from 1600 to 2000 in the past 4 years, what will it be 10 years hence? *Ans.* 3494.

14. The same as Problem 11 for 12 lbs. becoming 9 lbs. in 2 hrs. Also find how much is transformed at the end of 4 hrs.

15. For a constant temperature, the rate of change of the pressure of the atmosphere with respect to the height above sea level is proportional to itself. Find the pressure at a height of 5000 feet if it is 11.8 lbs. at 6000 feet and 15 lbs. at sea level. *Ans.* 12.28 lbs.

155. The Area under a Curve. Let RS, Fig. 152, be a portion of the curve whose equation may be represented by $y = f(x)$.

Let DB be an ordinate at a fixed distance a from the y axis and let MP be the ordinate of a variable point P on the curve. The area bounded by $DMPB$ is represented by A. *It is evident*

that the value of A depends upon the position of the variable ordinate MP and hence is a function of x.

Now let P move to Q so that x increases an amount Δx and A increases the corresponding amount ΔA or $MNQP$. The value of ΔA is such that

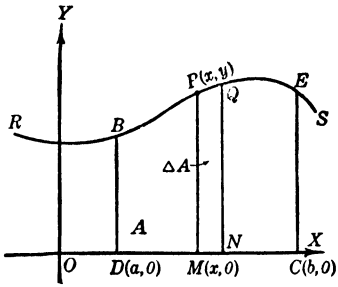


FIG. 152

(1) $\quad MP \cdot \Delta x < \Delta A < NQ \cdot \Delta x$

if the curve rises from P to Q. The inequality signs will merely be reversed if the curve falls from P to Q.

Dividing relation (1) by Δx, we have

$$MP < \frac{\Delta A}{\Delta x} < NQ.$$

Now let $\Delta x \to 0$ so that the ordinate NQ approaches MP in position and value, then, since $\Delta A/\Delta x$ remains between the two, it follows that

$$\text{(2)} \qquad \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} \equiv \frac{dA}{dx} = MP = y = f(x), \qquad (\S\, 52,\ \text{IV}).$$

That is, the rate of change of A with respect to x is the ***ordinate*** of the bounding curve at the extreme right-hand boundary. It is convenient to think of A as generated by the line-segment MP as it moves from the initial position DB determined by $x = a$ to some final position.

Relation (2) gives us the differential equation

$$dA = y\,dx,$$

and by integration we find

$$\text{(3)} \qquad A = \int y\,dx = \int f(x)\,dx = F(x) + C.$$

The expression $\int y\,dx$ of course has a meaning when either y is expressed as a function of x or when dx is replaced by its value in y and dy. The variable whose differential occurs under the integral sign is called the ***variable of integration;*** *if the integral has limits, they refer to the variable of integration.* Any other variables appearing in the integrand must be expressed in terms of the variable of integration before the integral can be evaluated.

The area between the two fixed ordinates DB and CE may now be evaluated. To determine C in (**3**), we have $x = a$ and $A = 0$ when P is at B. Hence $C = -F(a)$, and $A = F(x) - F(a)$. When P is at E, $x = b$ and $A = F(b) - F(a)$. Therefore (§ 148)

$$(4) \qquad A = \int_a^b f(x)\,dx = F(b) - F(a).$$

Similarly, the area between the curve $x = g(y)$, the y axis, and the lines $y = c$ and $y = d$ is

$$(5) \qquad A = \int_c^d g(y)dy.$$

The formulas (**4**) and (**5**) are subject to the definite restriction that the curve $y = f(x)$ does not cross the x axis, and the curve $x = g(y)$ does not cross the y axis between the limits of integration. The reason for this is explained at the close of the following article.

156. The Area between Two Intersecting Curves. Consider the area A between the two curves MN, $y = f_1(x)$, and RS, $y = f_2(x)$, from their intersection at $x = a$ to the variable vertical line P_1P_2. If the corresponding areas between each curve and the x axis are A_1 and A_2, then

$$(1) \qquad A = A_2 - A_1,$$

and

$$\frac{dA}{dx} = \frac{dA_2}{dx} - \frac{dA_1}{dx},$$

or

$$(2) \qquad \frac{dA}{dx} = y_2 - y_1 = f_2(x) - f_1(x).$$

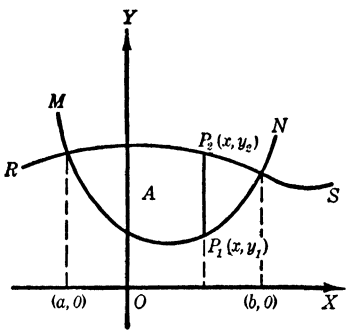


FIG. 153

Therefore the rate of change of A with respect to x is the vertical boundary P_1P_2. From (2) we have

$$(3) \qquad dA = [f_2(x) - f_1(x)]\,dx$$

and hence the area between the intersections at $x = a$ and $x = b$ is

$$(4) \qquad A = \int_a^b [f_2(x) - f_1(x)]\,dx.$$

The student should set up and use formula (4) in such problems instead of finding the area between each curve and the x axis.

If two curves intersect in more than two points, the limits of (4) refer to any two *consecutive intersections.* The reason for this is evident if we consider the two curves of Fig. 154. Suppose the curves $y = f_1(x)$ and $y = f_2(x)$ intersect in the three points B, C, and D. Then the area A_1 bounded by the two curves between B and C is generated by the line P_1P_2, which is $f_2(x) - f_1(x)$ in length. But the area A_2 between C and D bounded by the two curves is generated by P_2P_1, which has the length $f_1(x) - f_2(x)$. Then

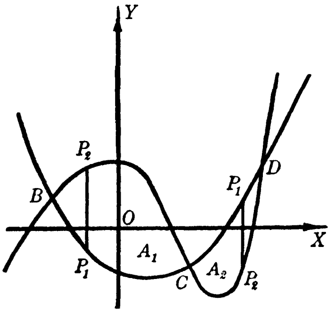


FIG. 154

$$dA_1 = [f_2(x) - f_1(x)]\,dx,$$

while

$$dA_2 = [f_1(x) - f_2(x)]\,dx.$$

These differentials differ in sign. If we integrate $[f_2(x) - f_1(x)]\,dx$ from point B to point D we get $A_1 - A_2$. That is, we get the algebraic sum of A_1 and $(-A_2)$ since $f_2(x) - f_1(x)$ is positive from B to C and negative from C to D.

EXAMPLES

1. Find the area between the curve $3y = x^3$ and the x axis from $x = -2$ to $x = 3$.

SOLUTION. The desired area is composed of the two parts BOC and ODE. (See Fig. 155.) The area BOC is generated by the line $P_1P_2 = -y = -x^3/3$ as P_2 moves from C to O. Hence the area of

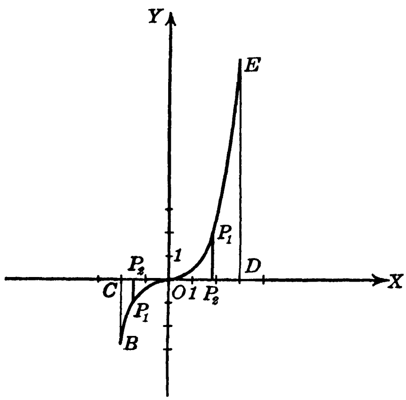


FIG. 155

$$BOC = \frac{1}{3}\int_{-2}^{0} (-x^3)\,dx = -x^4/12\Big]_{-2}^{0} = 4/3 \text{ sq. units.}$$

Similarly ODE is generated by $P_2P_1 = y = x^3/3$ and its value is

$$\frac{1}{3}\int_{0}^{3} x^3\,dx = x^4/12\Big]_{0}^{3} = 81/12 = 6\tfrac{3}{4} \text{ sq. units.}$$

Therefore the desired area is $8\tfrac{1}{12}$ square units.

2. Find the area bounded by the two curves $y = x^2$ and $y^2 = x$.

SOLUTION. The desired area is generated by P_1P_2 as it moves from O to B. Hence

$$dA = (y_2 - y_1)\,dx = (x^{1/2} - x^2)\,dx.$$

Therefore

$$A = \int_0^1 (x^{1/2} - x^2)\,dx$$

$$= \left(\frac{2\,x^{3/2}}{3} - \frac{x^3}{3}\right)\Bigg]_0^1 = \frac{1}{3} \text{ sq. units.}$$

A figure is an essential part of the solution of any area problem.

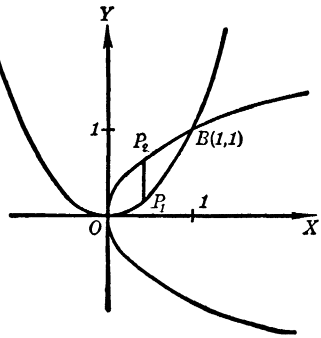


FIG. 156

PROBLEMS

1. Find the area between the lines $y = 3\,x$, $y = 0$, and $x = 6$ and check by the use of some formula. *Ans.* 54 sq. units.

2. Find the area under $y = 2\,x - x^2$ and above $y = 0$.

3. Given the lines $x + y = 0$ and $2\,x - y - 2 = 0$. Express any vertical line-segment from the first line to the second as a function of x. *Ans.* $3\,x - 2$.

4. In Problem 3, express any horizontal line-segment from the second line to the first as a function of y.

5. What is the length of a horizontal section of the area enclosed by $y^2 = 4\,x + 4$ and $x^2 = 4\,y + 8$. *Ans.* $2\sqrt{y + 2} + 1 - y^2/4$.

6. Find in terms of x the length of a vertical section of the area enclosed by $y^2 = 4\,x$ and $x = 2 + y$.

Find the area enclosed by the following boundaries. (Nos. 7–20.)

7. $y = x^2,\ y = 0,\ x = 1,\ x = 4.$ *Ans.* 21 sq. units.

8. $y = x^2 + 1,\ x + y = 1.$

9. $x = y^2 - 4,\ x = 0.$ *Ans.* 32/3 sq. units.

10. $x = y^2 + 2\,y,\ x + 2\,y = 0.$

11. $y = 5\,x - x^2,\ 2\,y = 5\,x - x^2.$ *Ans.* 125/12 sq. units

12. $y^2 = x + 6,\ x = 4 - y^2.$

13. $x = 2\cos\theta,\ y = 2\sin\theta.$ *Ans.* $4\,\pi$ sq. units.

14. $x = 4\cos\theta,\ y = 3\sin\theta.$

15. $y = e^{-x/2}$, $x = -2$, $x = 4$. *Ans.* $2(e - 1/e^2)$ sq. units.

16. $y = e^{-x}$, $y = e^x$, $x = 2$.

17. $y = \log x$, $x = 0$, $y = 0$, $y = 2$. *Ans.* $e^2 - 1$ sq. units.

18. $y = x \log x$, $y = x - x^2$.

19. $x = \sin y$, $y = 0$, $x = 1$. *Ans.* $\pi/2 - 1$ sq. units.

20. $y = \sin x$, $y = -1$, $x = 0$, $x = \pi$.

Find the area enclosed by the loops of the following curves. (Nos. 21–23.)

21. $y^2 = x^2(x + 2)$. *Ans.* $(32/15)\sqrt{2}$ sq. units.

22. $y^2 = (x - 1)(x - 3)^2$.

23. $x^2 = y^4(y + 4)$. *Ans.* $4096/105$ sq. units.

Find the area bounded by the following curves. (Nos. 24–27.)

24. The ellipse $x^2/a^2 + y^2/b^2 = 1$.

25. An arch of $y = \sin^2(2x)$ and the x axis. *Ans.* $\pi/4$ sq. units.

26. The coordinate axes and the parabola $x^{1/2} + y^{1/2} = a^{1/2}$.

27. The equilateral hyperbola $x^2 - y^2 = a^2$ and the double ordinate through the point P_1 on the curve. *Ans.* $x_1y_1 - a^2 \log [(x_1 + y_1)/a]$.

28. Prove that the area of a parabolic segment formed by a line-segment perpendicular to its axis is two-thirds of the rectangle which circumscribes the segment.

157. Some Improper Integrals.

A definite integral whose integrand becomes infinite for some value or values of the variable of integration in the interval of integration or which has an infinite

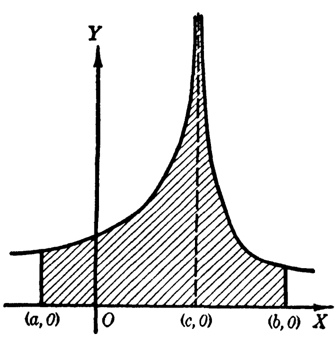


FIG. 157

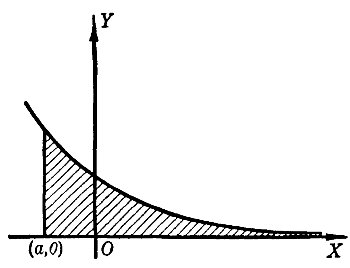


FIG. 158

limit is called an ***improper integral.*** If we consider the geometric interpretation of such an integral, the reason for this is at once evident. The definite integral, *i.e.*, $\int_a^b f(x)dx$, represents geo-

metrically the number of square units in that part of the xy plane which is ***bounded*** by $y = f(x)$, $x = a$, $x = b$, and the x axis, and this exists for all continuous functions $f(x)$ in the interval $a \leqq x \leqq b$. If, however, the integrand becomes infinite for some x between $x = a$ and $x = b$, the curve $y = f(x)$ has a vertical asymptote and the area represented by the integral has its boundary recede to infinity and it is *no longer bounded.* Figure 157 shows such an unbounded area due to the asymptote at $x = c$.

Again, if a limit of the integral is infinity, the bounding curve has a horizontal asymptote, thereby making the area unbounded also. Figure 158 shows such an area. Such improper integrals are defined as follows:

(*a*) *If $f(x)$ becomes infinite at one of the limits of integration, say at $x = b$, then*

$$(1) \qquad \int_a^b f(x)\,dx \equiv \lim_{h \to 0} \int_a^{b-h} f(x)\,dx, \quad h > 0.$$

(*b*) *If $f(x)$ becomes infinite at $x = c$ where $a < c < b$, then*

$$(2) \qquad \int_a^b f(x)\,dx \equiv \lim_{h_1 \to 0} \int_a^{c-h_1} f(x)\,dx + \lim_{h_2 \to 0} \int_{c+h_2}^b f(x)\,dx,$$

where h_1 and h_2 are positive.

(*c*) *If one limit is infinite, say the upper one, then*

$$(3) \qquad \int_a^\infty f(x)\,dx \equiv \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

These three definitions and the methods of evaluation of such integrals are illustrated by the following examples.

EXAMPLES

1. Find the area under the curve $y = x^{-1/2}$ from $x = 0$ to $x = 2$.

SOLUTION. The curve $y = x^{-1/2}$ has $x = 0$ as a vertical asymptote. The method is then to find the value of the shaded area in Fig. 159 and evaluate the limit of that area as $h \to 0$, if possible.

$$A = \int_0^2 x^{-1/2}\,dx = \lim_{h \to 0} \int_h^2 x^{-1/2}\,dx = \lim_{h \to 0} \left(2\,x^{1/2} \Big]_h^2 \right)$$

$$= \lim_{h \to 0} (2\sqrt{2} - 2\sqrt{h}) = 2\sqrt{2} \text{ sq. units.}$$

At first the student might think that the area under such a curve is always

infinite since its boundary recedes to infinity along the y axis. But for this curve such is evidently not true and it is essential to evaluate the integral in each case before a conclusion is reached.

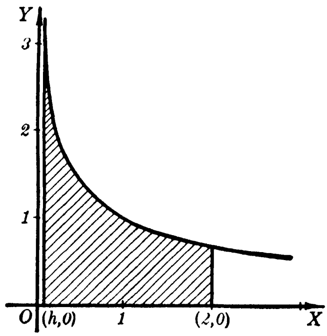


FIG. 159

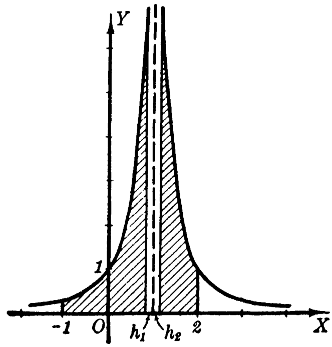


FIG. 160

2. Evaluate if possible $\int_{-1}^{2} \frac{dx}{(x-1)^2}$.

SOLUTION. The graph of the curve represented by $y = 1/(x-1)^2$ is shown in Fig. 160. It has a vertical asymptote, $x = 1$. Hence

$$\int_{-1}^{2} \frac{dx}{(x-1)^2} = \lim_{h_1 \to 0} \int_{-1}^{1-h_1} \frac{dx}{(x-1)^2} + \lim_{h_2 \to 0} \int_{1+h_2}^{2} \frac{dx}{(x-1)^2}, \quad h_1, h_2 > 0,$$

$$= \lim_{h_1 \to 0} \left(-\frac{1}{x-1} \Big]_{-1}^{1-h_1} \right) + \lim_{h_2 \to 0} \left(-\frac{1}{x-1} \Big]_{1+h_2}^{2} \right)$$

$$= \lim_{h_1 \to 0} \left(\frac{1}{h_1} - \frac{1}{2} \right) + \lim_{h_2 \to 0} \left(-1 + \frac{1}{h_2} \right)$$

$$= \infty,$$

since $1/h_1$ and $1/h_2 \to +\infty$ as h_1 and $h_2 \to 0$. In this case the *area under the curve has no meaning.*

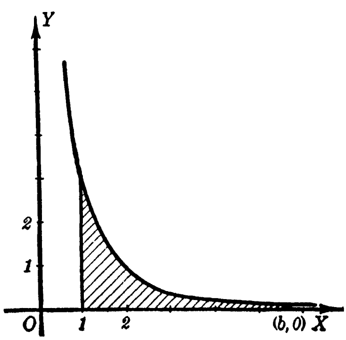


FIG. 161

3. Find the area under the curve $y = 3x^{-2}$ from $x = 1$ to $x = \infty$.

SOLUTION. As this example involves a horizontal asymptote, we evaluate the shaded area in Fig. 161 and then find the limit of such an area as $b \to \infty$, if possible. Thus

$$\int_{1}^{\infty} 3x^{-2}dx = \lim_{b \to \infty} \int_{1}^{b} 3x^{-2}dx$$

$$= 3 \lim_{b \to \infty} \left(-\frac{1}{x} \Big]_{1}^{b} \right)$$

$$= 3 \lim_{b \to \infty} \left(-\frac{1}{b} + 1 \right) = 3,$$

since $1/b \to 0$ as $b \to \infty$. Hence this area exists and is 3 square units. *A figure is very important in all such problems as it shows the places where the area is not bounded.*

PROBLEMS

Evaluate, if possible, each of the following improper integrals. (Nos. 1–14.)

1. $\int_0^2 dx/\sqrt{4 - 2x}$. *Ans.* 2.

2. $\int_1^2 dx/(2 - x)^{2/3}$.

3. $\int_0^3 dx/\sqrt{3 - x}$. *Ans.* $2\sqrt{3}$.

4. $\int_0^1 (e^x + 1)\,dx/(e^x - 1)$.

5. $\int_{-1/3}^{2/3} dx/\sqrt{4 - 9x^2}$. *Ans.* $2\pi/9$.

6. $\int_0^3 dx/(2x - 1)^2$.

7. $\int_1^4 dx/(2x - 3)^{2/3}$. *Ans.* $(3/2)(\sqrt[3]{5} + 1)$.

8. $\int_0^3 dx/(3 - x)^2$.

9. $\int_0^1 x^3\,dx/(x^2 - 1)$. *Ans.* $-\infty$.

10. $\int_0^3 2x\,dx/(x^2 - 1)^{4/3}$.

11. $\int_{-2}^{\sqrt{3}} dx/(4 - x^2)$. *Ans.* ∞.

12. $\int_0^4 dx/(3 - 5x)^{5/3}$.

13. $\int_0^1 dx/x(1 + x^2)$. *Ans.* ∞.

14. $\int_0^1 \sqrt{1 + x^2}\,dx/x$.

See whether the curves below define definite areas, and if so, find the area.

15. $y^2(4 - x) = x^2,\ x = 4,\ y = 0$. *Ans.* 32/3 sq. units.

16. $y(x - 2)^{2/3} = 1,\ x = 0,\ x = 4,\ y = 0$.

17. $y(x - 4)^3 = 8,\ y = 0,\ x = 1,\ x = 4$. *Ans.* No.

18. $y(x - 2)^2 = 3,\ y = 0,\ x = 2,\ x = 4$.

19. $y(1 - x) = 1,\ x = 0,\ x = 1,\ y = 0$. *Ans.* No.

20. $y(1 - x^2)^{3/2} = x,\ x = 0,\ x = 1,\ y = 0$.

21. $y^3(x - 1)^2 = 8x^3,\ y = 0,\ x = 0,\ x = 3$. *Ans.* $9[\sqrt[3]{2} + 1/2]$ sq. units.

22. $xy^2(1 + x^2) = 2,\ y = 0,\ x = 0,\ x = 4$.

23. $y = \log x,\ x = 0,\ y = 0$. *Ans.* 1 sq. unit.

24. The cissoid $y^2 = x^3/(2a - x)$ and its asymptote $x = 2a$.

158. Volume Generated by a Portion of a Plane. We have seen that the definite integral $\int_a^b f(x)dx$ may be interpreted as the area of a portion of the xy plane which is generated by a

vertical line-segment, one end of the line-segment remains on the curve $y = f(x)$ and the other on the x axis, as it moves from $x = a$ to $x = b$.

Now suppose the form of a system of parallel cross-sections of a solid is known and that the area of any one of the sections can be expressed in terms of the distance of the section from some fixed point of the solid. We may then consider such a solid as generated by a plane cross-section as it moves throughout the length of the solid.

EXAMPLE

The cross-sections of a certain solid are squares whose sides are equal to the squares of the distances of their planes from one end of the solid. Find the volume of the solid if it is 8 inches long.

SOLUTION. Take the origin as the pointed end of the solid and assume the cross-sections perpendicular to the x axis. Consider the portion of the solid between two cross-sections of distances x and $x + \Delta x$ from O. It is evident, since the solid increases in cross-section as x increases, that

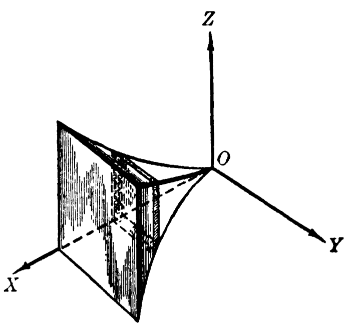


FIG. 162

$$x^4 \cdot \Delta x < \Delta V < (x + \Delta x)^4 \cdot \Delta x,$$

where ΔV is the part of the solid between the two cross-sections.

Dividing through by Δx we have

$$x^4 < \frac{\Delta V}{\Delta x} < (x + \Delta x)^4.$$

Now let $\Delta x \to 0$; we conclude that

$$\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} \equiv \frac{dV}{dx} = x^4. \qquad (\S\ 52,\ \text{IV.})$$

Therefore $dV = x^4\,dx$; and, integrating between the limits $x = 0$ and $x = 8$, we have

$$V = \int_0^8 x^4\,dx = \frac{x^5}{5}\Big]_0^8 = 6553\tfrac{3}{5} \text{ cubic inches.}$$

PROBLEMS

1. Planes perpendicular to the x axis cut a solid in circular sections with diameters joining $y = x$ and $y = 2x$. Find the volume of the solid between the planes $x = 0$ and $x = 5$. *Ans.* $125\,\pi/12$ cu. units.

2. Planes perpendicular to the x axis cut circles with diameters from $y = 1$ to $x^2 = 5 - y$ from a solid. What is the volume between the intersections of the line and parabola?

3. A variable square moves with its plane perpendicular to the y axis and with the ends of one diagonal on the ellipse $4x^2 + y^2 = 16$. Find the volume generated. *Ans.* 128/3 cu. units.

4. Plane sections of a solid perpendicular to the x axis are squares with the ends of a diagonal on $16y = x^2$ and $4y = x^2 - 12$, respectively. Find the volume between the intersections of the curves.

5. A variable square moves with its plane perpendicular to the y axis and with the ends of one side on $y = x$ and $y^2 = 4x$, respectively. What volume is generated between the intersections of the line and the parabola? *Ans.* 32/15 cu. units.

6. A variable equilateral triangle moves with its plane perpendicular to the x axis. One side extends in the xy plane between the parabolas $y^2 = 4x$ and $y^2 = 16x$. What volume is generated from the origin to $x = 2$?

7. An equilateral triangle moves so that its plane is perpendicular to the y axis. One end of its base is on the y axis and the other end on the circle $x^2 + y^2 = a^2$. What volume is generated? *Ans.* $(2/3)\,a^3\sqrt{3}$ cu. units.

8. Sections of a solid perpendicular to the x axis are squares with a side the double ordinate of $x^2 + 4y^2 = 4$. What volume lies between $x = -2$ and $x = 2$?

9. In Problem 8, use the y axis, double abscissas, and $y = 0$ to $y = 1$. *Ans.* 32/3 cu. units.

10. The base of a solid may be represented by the area bounded by $y^2 = 6x$ and $y^2 = 6 - 3x$. Cross-sections by planes perpendicular to the y axis are equilateral triangles. Find the volume of the solid.

11. The same as Problem 10, except that the sections are squares. *Ans.* 128/15 cu. units.

12. The same as Problem 10, except the sections are isosceles triangles with altitudes the same as their bases.

13. Sections of a solid perpendicular to some line are squares whose sides are the square roots of the distances to the sections from one end of the line. Find the volume if the line is 6 units long. *Ans.* 18 cu. units.

14. A variable circle moves with its plane perpendicular to the x axis, its center on $y = \sin 2x$ and an end of a diameter on the x axis. What volume is determined by one arch of the sine curve?

15. A chip of constant angle θ is cut from a tree of radius r feet by a horizontal saw-cut and an oblique axe-cut. What is the volume of the chip if it extends halfway through the tree? *Ans.* $(2/3)r^3 \tan\theta$ cu. ft.

ADDITIONAL PROBLEMS

1. A curve passes through the point (0, 1) and its slope at any point is $3 - 5x - 3x^2$. Find the equation of the curve.

Ans. $2y = 2 + 6x - 5x^2 - 2x^3$.

2. Find the equation of the curve through the origin whose slope at any point is $x/(1 - x^2)$.

3. A curve of slope $\cos 3x$ passes through the point $(\pi/3, 2)$. What is its equation? *Ans.* $3y = \sin 3x + 6$.

4. Find the equation of the curve whose slope is xy^2 if it passes through the point (4, 3).

5. Find the equation of the curve whose rate of change of slope with respect to x is $2 + 4x$, if the curve has a low point at (1, 3).

Ans. $3y = 3x^2 + 2x^3 - 12x + 16$.

6. Find the equation of the curve through $(1, -6)$ which intersects the curves $2y = \log c(1 + x^2)$ at right angles.

7. An object moving in a straight line has an acceleration proportional to the square root of its velocity. It has an initial velocity of 16 feet per second and comes to rest after 6 seconds. How far does it move? *Ans.* 32 ft.

8. If the acceleration of a particle along a straight line is $2/v$ and its velocity is 3 units/sec. when $t = 0$, how far does it move during the next 4 seconds?

9. If the acceleration of a body is proportional to its velocity, find an expression for the distance traveled if $s = 0$ and $v = 1$ unit/sec. when $t = 0$.

Ans. $s = (1/k)(e^{kt} - 1)$ units.

10. A man on a bank of a river 60 ft. above the water can throw a stone with initial velocity of 100 ft./sec. What angles of elevation should be used to hit a spot on the water 125 ft. from the bank?

11. An investment of \$4000 is depreciating at the constant rate of 4% per year. When will the investment be worth \$3000? *Ans.* In 7.19 years.

12. For a constant temperature, the rate of change of the pressure of the atmosphere with respect to the height above sea level is proportional to itself. Find the pressure at a height of 4500 ft. if it is 11.9 lbs. at 6000 ft. and 15 lbs. at sea level.

13. Find the equation of the paths of a point whose rate of change of ordinate with respect to its abscissa is proportional to the ordinate. Which one of the curves passes through (0, 2) with the slope 4? *Ans.* $y = 2e^{2x}$.

14. If the rate of change of the slope of a curve with respect to the ordinate at any point is proportional to the slope, find the path of the point.

Find each of the areas bounded as follows. (Nos. 15–20.)

15. $y = 6x - x^2$, $y = 0$. *Ans.* 36 sq. units.

16. $y = x^2$, $y = 6 - x^2$.

17. $x = 4 - y^2$, $x = y^2 + 1$. *Ans.* $2\sqrt{6}$ sq. units.

18. $4y = x^2$, $4y = x^3$, $x = 2$.

19. $y^2 = 2px$ and its latus rectum. *Ans.* $(2/3)p^2$ sq. units.

20. The loop of $y^2 = x^3 - x^4$.

Evaluate, if possible, each of the following improper integrals. (Nos. 21–26.)

21. $\int_0^1 dx/(2x - 1)^2$. *Ans.* ∞.

22. $\int_{\sqrt{5}}^4 x^3\,dx/\sqrt{5 - x^2}$.

23. $\int_0^4 dx/(3 - 2x)^{2/3}$. *Ans.* 4.738.

24. $\int_0^2 x^3\,dx/(1 - 2x^2)^{4/3}$.

25. $\int_{-1}^0 x\,dx/(1 + x)^{3/2}$. *Ans.* $-\infty$.

26. $\int_0^{\pi/2} \sec^4\theta\,d\theta$.

27. The cross-section of a solid perpendicular to a line AB is a right triangle with its hypotenuse equal to the distance of the section from A. If AB is 4 units and one acute angle of the section is always $\pi/6$, find the volume of the solid. *Ans.* $8\sqrt{3}/3$ cu. units.

28. Sections of a solid perpendicular to the x axis are squares with sides as double ordinates of $y^2 = 4x$. What is the volume between $x = 0$ and $x = 2$?

29. A variable circle moves with its plane perpendicular to the x axis and a diameter extends from $16y = x^2$ to $4y = x^2 - 12$. What volume is there between the intersections of the curves? *Ans.* 9.6π cu. units.

30. A variable circle moves with its plane perpendicular to the y axis. It passes through the y axis and its center is on the ellipse $x^2/a^2 + y^2/b^2 = 1$. What volume is generated?

31. A trough 10 ft. long has cross-sections which are parabolic segments of altitude and base each 4 ft. What is its volume? *Ans.* 320/3 cu. ft.

32. The retarding effect of fluid friction on a rotating disc is proportional to the angular velocity of the disc. Find an expression for the angular velocity at any time.

33. A pair of shears is wanted which has a constant cutting angle. What equation may represent the edge of one blade? *Ans.* $r = ce^{k\theta}$.

34. Water flows out of a hemispherical basin through a hole at the bottom in such a manner that the volume of the water at any time decreases at a rate proportional to the square root of the depth of the water remaining in the basin. What volume remains at any time and at what depth does the surface of the water fall most slowly? *Ans.* $(\pi h^2/3)(3r - h)$ cu. units; $h = 2r/3$.

CHAPTER XIII

THE DEFINITE INTEGRAL AS THE LIMIT OF A SUM

159. An Approximation of the Value of a Definite Integral. A function, whose rate of change is known, has been found by reversing the operation of differentiation. If this reversal is inconvenient or impossible, an approximation of any desired accuracy may be found for the value of the function or integral if it is a definite integral. This can be done because we have seen that any proper integral may be represented geometrically by a definitely bounded area.

Suppose the value of the proper integral $\int_a^b f(x)\,dx$ is desired. If we sketch the curve $y = f(x)$, as shown in Fig. 163, the numerical value of the integral is the number of square units in the area $MNPQ$, regardless of what geometric or physical meaning the integral may have. Obviously this area may be approximated by finding the sum of the areas of a system of rectangles either inscribed under the curve or circumscribed over it. In constructing the rectangles shown in Fig. 163, we take their width Δx as any convenient divisor of the interval MN. It appears that the resulting approximation becomes better as we decrease the width of the rectangles and consequently increase their number.

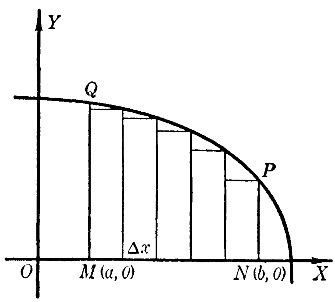


FIG. 163

If the interval $b - a$ is divided into n equal parts, $b = a + n\cdot\Delta x$. The lengths of the n rectangles will then be either $f(a), f(a + \Delta x), \cdots, f[a + (n - 1)\,\Delta x]$, or $f(a + \Delta x), f(a + 2\,\Delta x), \cdots, f(b)$.

EXAMPLE

Find several approximations for $\int_1^4 (26 - x^2)dx$. Note that they become better as the number of rectangles is increased.

SOLUTION. The curve over the area to be evaluated is the parabola $y = 26 - x^2$. The three rectangles shown in the figure have a combined area of $(22 + 17 + 10)$, which is 49 units. Now suppose six rectangles under the curve from $x = 1$ to $x = 4$. Their width is $1/2$, their heights are 23.75, 22, 19.75, 17, 13.75, 10 respectively, and their combined area is 53.125 units. Such treatment may be continued until any desired accuracy is acquired. Thus it appears that the value of the given integral is the limit approached by such sums as the width Δx of each rectangle approaches zero as a limit. The exact value is

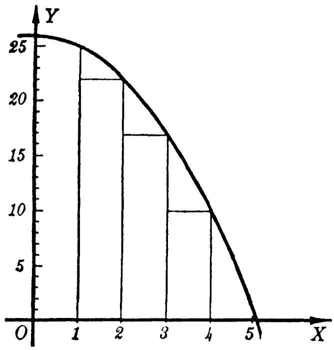


FIG. 164

$$\int_1^4 (26 - x^2)dx = (26\,x - x^3/3)\Big]_1^4$$
$$= 57 \text{ units.}$$

We shall show in the next article that, if the curve $y = f(x)$ is *continuous* in the interval $a \leqq x \leqq b$, the value of the definite integral $\int_a^b f(x)dx$ is such a limit.

PROBLEMS

Draw a figure representing each of the following integrals and use $\Delta x = 0.5$ unit to approximate the value of each. Either inscribed or circumscribed rectangles may be used. The figure should help you decide which to use.

1. $\int_{-1}^{2} \sqrt{x + 3}\, dx$. *Ans.* 5.77 units.

2. $\int_0^2 \sqrt{4 + x^2}\, dx$.

3. $\int_3^5 dx/\sqrt{1 + x^2}$. *Ans.* 0.465 unit.

4. $\int_2^5 dx/\sqrt{4\,x^2 - 5}$.

5. $\int_{-2}^{0} dx/(2\,x^2 + 3)^{1/2}$. *Ans.* 1.104 units.

6. $\int_{-2}^{2} (4\,x^2 + 1)^{1/3}\, dx$.

7. $\int_{-1}^{2} (1 + x^2)^{2/3}\, dx$. *Ans.* 4.34 units.

8. $\int_{-2}^{1} (2 - x^3)^{3/2}\, dx$.

160. The Definite Integral as the Limit of a Sum. Since in most applications the meaning of the definite integral is something very different from an area, it is desirable to prove the following analytic theorem without any reference to its possible geometric interpretation. The importance of the theorem in its wide range of applications is such that it is generally called the ***fundamental theorem of the integral calculus.***

THEOREM. *Let $F(x)$ and its derivative $f(x)$ be continuous functions of x in the interval $a \leqq x \leqq b$. Divide this interval into n sub-intervals $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$, where $\Delta x_i > 0$.*

Let x_i be any value of the variable in the corresponding sub-interval Δx_i. Then the limit of the sum of differential products

$$\sum_{i=1}^{n} f(x_i)\Delta x_i$$

as n increases indefinitely and each Δx_i decreases and approaches zero, is the value of the definite integral

$$\int_a^b f(x)\,dx \equiv F(b) - F(a).$$

In symbols this conclusion is written

$$\lim_{\substack{n \to \infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx = F(b) - F(a).$$

PROOF. Let the values of the variable for the points of subdivision be $a_1, a_2, a_3, \cdots, a_{n-1}$. That is, $a + \Delta x_1 = a_1, a_1 + \Delta x_2 = a_2, \cdots, a_{n-1} + \Delta x_n = b$. Then by the Theorem of Mean Value, § 90, we have

$$F(x + \Delta x) - F(x) = f(\bar{x}) \cdot \Delta x,$$

for some $\bar{x}$ in the interval Δx. Or, since $f(x)$ is a continuous function,

$$F(x + \Delta x) - F(x) = [f(x') + e]\Delta x$$

for any x' in the interval Δx, where $\lim\limits_{\Delta x \to 0} e = 0$.

Applying this theorem successively to each of the sub-intervals, we have the relations

$$
\begin{aligned}
F(a_1) - F(a) &= f(x_1)\cdot\Delta x_1 + e_1\cdot\Delta x_1.\\
F(a_2) - F(a_1) &= f(x_2)\cdot\Delta x_2 + e_2\cdot\Delta x_2.\\
F(a_3) - F(a_2) &= f(x_3)\cdot\Delta x_3 + e_3\cdot\Delta x_3.\\
\cdot \qquad & \qquad \cdot \qquad \cdot\\
\cdot \qquad & \qquad \cdot \qquad \cdot\\
F(a_{n-1}) - F(a_{n-2}) &= f(x_{n-1})\cdot\Delta x_{n-1} + e_{n-1}\cdot\Delta x_{n-1}.\\
F(b) - F(a_{n-1}) &= f(x_n)\cdot\Delta x_n + e_n\cdot\Delta x_n.
\end{aligned}
$$

The sums of corresponding members of these give, for any number n,

$$F(b) - F(a) = \sum_{i=1}^{n} f(x_i)\cdot\Delta x_i + \sum_{i=1}^{n} e_i\cdot\Delta x_i.$$

Now let n increase indefinitely by making each Δx_i approach zero. Each change in the value of n produces a new set of e's such that* $\lim\limits_{\Delta x_i\to 0} e_i = 0$. Then we have

$$F(b) - F(a) = \lim_{\substack{n\to\infty\\ \Delta x_i\to 0}} \left[\sum_{i=1}^{n} f(x_i)\cdot\Delta x_i + \sum_{i=1}^{n} e_i\cdot\Delta x_i\right]\cdot$$

Now consider the second sum of the right-hand member. Among the e's there is an e_k, such that $|e_k|$ for a given n is greater than or at least equal to any other $|e_i|$. Then since

$$\sum_{i=1}^{n} e_i\cdot\Delta x_i = e_1\cdot\Delta x + e_2\cdot\Delta x_2 + \cdots + e_n\cdot\Delta x_n,$$

$$\left|\sum_{i=1}^{n} e_i\cdot\Delta x_i\right| \leqq \sum_{i=1}^{n} |e_i|\cdot\Delta x_i \leqq |e_k|\sum_{i=1}^{n}\Delta x_i.$$

But by hypothesis $\sum\limits_{i=1}^{n}\Delta x_i = b - a$ for all values of n and $e_i\to 0$ as $\Delta x_i\to 0$. Hence

$$\lim_{\substack{n\to\infty\\ \Delta x_i\to 0}} \left|\sum_{i=1}^{n} e_i\cdot\Delta x_i\right| \leqq \lim_{\substack{n\to\infty\\ \Delta x_i\to 0}} [e_k(b-a)] = 0,$$

if the interval $b - a$ is finite. Therefore

$$F(b) - F(a) \equiv \int_a^b f(x)dx = \lim_{\substack{n\to\infty\\ \Delta x_i\to 0}} \sum_{i=1}^{n} f(x_i)\cdot\Delta x_i.$$

* This is due to a theorem in a more advanced course in which such a function as $f(x)$ is known as *uniformly continuous.*

When the interval $b - a$ is not finite, the result is an improper integral which has been treated in the preceding chapter.

The quantities Δx_i and e_i are ***infinitesimals,*** *being variables each of whose limits is zero.* If an infinitesimal Δx_i is multiplied by some finite number not zero, say $f(x_i)$, the product is still an infinitesimal and the two infinitesimals $f(x_i) \cdot \Delta x_i$ and Δx_i are of the same order. But the products $e_i \cdot \Delta x_i$ of two infinitesimals are infinitesimals of ***higher order than either e_i or Δx_i*** (§ 85).

Since the $\lim\limits_{\substack{n \to \infty \\ \Delta x_i \to 0}} \sum\limits_{i=1}^{n} e_i \cdot \Delta x_i$ was proved to be zero, these infinitesimals $e_i \cdot \Delta x_i$ of higher order than the first could have been omitted without affecting the value of the limit of the sum which gives the integral. This is of very great importance because often the ***element of integration*** cannot be expressed exactly as $f(x_i) \cdot \Delta x_i$, but includes some additional infinitesimal of higher order. However, such infinitesimals may be discarded, since the limit of their sum as the number increases indefinitely is zero in all cases with which we shall deal.

In applications of the fundamental theorem, the expression for the element of integration is simplified if the Δx_i are taken equal and if the x_i are chosen as divisional values of the variable. As this choice is possible under the general proof given above, we shall use it in illustrations.

161. Geometric Illustration of the Fundamental Theorem. Suppose AB is the graph of the continuous curve $y = f(x)$ from $x = a$ to $x = b$. The rectangles of width Δx under the arc have as the sum of their areas

$$(1) \quad f(a) \cdot \Delta x + f(a + \Delta x) \cdot \Delta x + f(a + 2\,\Delta x) \cdot \Delta x + \cdots + f(b - \Delta x) \cdot \Delta x,$$

a quantity which is less than the area under the arc.

The rectangles over the arc of the curve have as the sum of their areas

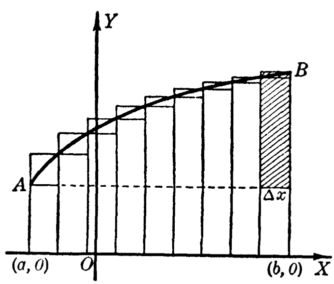


FIG. 165

$$(2) \qquad f(a + \Delta x) \cdot \Delta x + f(a + 2\,\Delta x) \cdot \Delta x + \cdots + f(b) \cdot \Delta x,$$

which quantity is greater than the area under the arc.

But these two sums differ in only the first term of (1) and the last term of (2). Their difference is then

$$(3) \qquad [f(b) - f(a)]\Delta x,$$

which is the area of the shaded rectangle of Fig. 165. Obviously, this difference approaches zero as a limit as $\Delta x \to 0$ and hence each sum approaches the area under the arc as a limit and again we have

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \Delta x\to 0}} \sum_{i=1}^{n} f(x_i)\cdot\Delta x.$$

In this discussion we have assumed that along the arc AB, as x increases from a to b, y increases continuously.

If y decreases as x increases, the same situation would occur except the rectangles under the curve would be represented by (2) above and those over the curve by (1). Finally, if the arc AB has a finite number of maxima and minima, the reasoning given above would be applied to each segment of arc along which y steadily increases or decreases as x increases.

162. Geometric Illustration Using Polar Coordinates. The same geometric reasoning may be applied if the integral is expressed in polar coordinates. For instance, the area OPQ may be approximated by the sum of a set of circular sectors of angle $\Delta\theta$ either inscribed in the arc PQ or circumscribed about it. One such sector has the area $r_i^2\cdot\Delta\theta/2$ and the difference of the two sums is evidently the shaded area, which is represented by

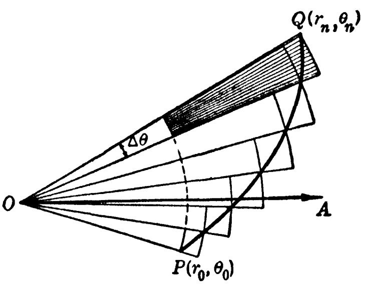


FIG. 166

$$\frac{1}{2}(r_n^2 - r_0^2)\Delta\theta.$$

This difference approaches zero as a limit when $\Delta\theta \to 0$ and hence the limit of either sum is the area of OPQ. We may write then

$$A = \lim_{\substack{n\to\infty \\ \Delta\theta\to 0}} \sum_{i=1}^{n} \frac{1}{2} r_i^2\cdot\Delta\theta = \int_{\theta_0}^{\theta_n} \frac{r^2}{2}\,d\theta.$$

Consequently, any integral of the form $\int_{\theta_0}^{\theta_n} f(\theta)d\theta$ may represent an area and is the limit of the sum of a set of circular sectors, if $f(\theta)$

is computed from the expression $r^2/2$, where $r = \phi(\theta)$ is the equation of the bounding curve, such as PQ.

Since any proper integral in polar coordinates may accordingly be interpreted as an area or has the numerical value of the square units of a definite area, the student should see that the theorem of the preceding article may have different types of illustration. Also, assumptions analogous to those made in the discussion involving x and y must be made here concerning continuity and variation of r and θ.

163. Areas. The idea of the definite integral as the limit of a sum permits us to *set up* the expression for the area bounded by given curves as the limit of the sum of ***elements*** of the area. Each *element is a rectangle* if the bounding curves are given in rectangular coordinates, a *sector of a circle* if the equations of the curves are expressed in polar coordinates. In either system of coordinates the student must be careful to choose, as *elements of the desired area, rectangles or sectors which have the same characteristics throughout the area, that is, every such element has its area represented by the same type of differential product.* In the case of rectangular coordinates this means that all rectangles needed for a good approximation of the desired area must have their corresponding ends touching the same curves. It may be necessary to use horizontal rectangles to bring this about. If all elements cannot be made of the ***same type,*** it is necessary to divide the area into parts over each of which ***typical elements*** may be used. In polar coordinates all sectors must reach from the pole to the same curve, and, if the element is the difference of two sectors, the corresponding ends of such elements must touch the same curves.

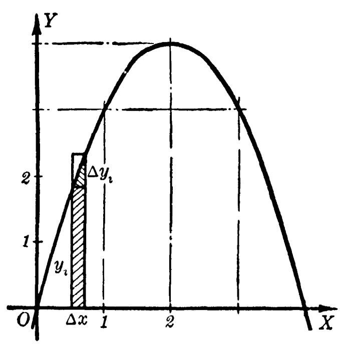


FIG. 167

EXAMPLES

1. Find the area between the curve $y = 4x - x^2$ and the x axis.

SOLUTION. A figure is an essential part of a solution of this type of problem. Solving for the intersections of $y = 4x - x^2$ and $y = 0$, we have as limits along the x axis, $x = 0$ and $x = 4$. A typical element shown by the shaded area is

$$\Delta A_i = y_i \cdot \Delta x + k_i \cdot \Delta y_i \cdot \Delta x,$$

where $0 < k_i < 1$ and $k_i \cdot \Delta y_i \cdot \Delta x$ of higher order than $y_i \cdot \Delta x$ may be discarded in taking the $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \Delta A_i$. Therefore

$$A = \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \sum_{i=1}^{n} (y_i \cdot \Delta x) = \int_0^4 y\,dx.$$

Then

$$A = \int_0^4 (4x - x^2)dx,$$

since y refers to the ordinate of the given parabola. Whence

$$A = \left(2x^2 - \frac{x^3}{3}\right)\Big]_0^4 = 32 - \frac{64}{3} = \frac{32}{3} \text{ square units.}$$

2. Find the area bounded by the two curves $y = x^3 - x^2$ and $y = x^2$.

Solution. Solving simultaneously, we find the intersections of the curves at $x = 0$, and $x = 2$. Let y_2 and y_1 represent any corresponding ordinates of the parabola and the cubic respectively. Then the shaded area is

$$\Delta A_i = (y_2 - y_1)_i \cdot \Delta x + (k_2 \cdot \Delta y_2)_i \cdot \Delta x - (k_1 \cdot \Delta y_1)_i \cdot \Delta x,$$

where both k_1 and k_2 lie between 0 and 1, and where $(k_2 \cdot \Delta y_2 - k_1 \cdot \Delta y_1)_i \cdot \Delta x$, of higher order than $(y_2 - y_1)_i \cdot \Delta x$, may be discarded in the limit giving A. Whence

$$A = \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \sum_{i=1}^{n} [(y_2 - y_1)_i \cdot \Delta x] = \int_0^2 (y_2 - y_1)dx.$$

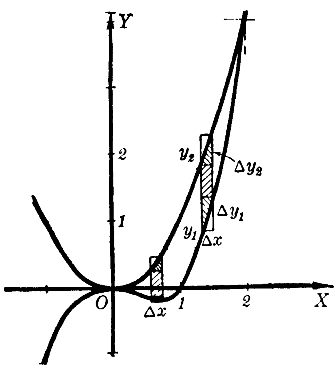


Fig. 168

Therefore

$$A = \int_0^2 [x^2 - (x^3 - x^2)]dx$$

$$= \int_0^2 (2x^2 - x^3)dx$$

$$= \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Big]_0^2$$

$$= \frac{16}{3} - \frac{16}{4} = \frac{4}{3} \text{ square units.}$$

The expression for the element of area ΔA_i should always be set up before integration and care must be taken to let y_1 represent an ordinate of one curve and y_2 the corresponding ordinate of the other curve. Then if read positively, as $y_2 - y_1$ in Fig. 168, and if the sum is taken in the positive direction along the x axis, the result of integration is always positive. *Such an element must be set up for each distinct part of a composite area.*

3. Find the area of the *cardioid* $r = a(1 + \cos\theta)$.

Solution. The shaded element of area is

$$\Delta A_i = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} k_i[r_i^2 - (r_i + \Delta r_i)^2]\Delta\theta, \quad 0 < k_i < 1,$$

where $(1/2)k_i(2\,r_i\Delta r_i + \overline{\Delta r_i}^2)\,\Delta\theta$ is of higher order than $(1/2)r_i^2\Delta\theta$. Whence

$$A = \lim_{\substack{n \to \infty \\ \Delta\theta \to 0}} \sum_{i=1}^{n} \left[\frac{1}{2} r_i^2 \cdot \Delta\theta\right] = 2 \int_0^{\pi} \frac{1}{2} r^2 d\theta.$$

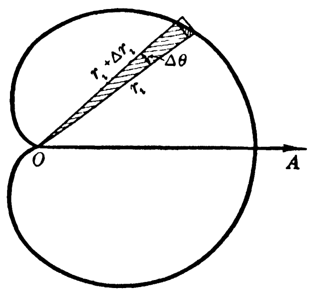


Fig. 169

Therefore

$$A = a^2 \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)d\theta$$

$$= a^2\left(\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big]_0^{\pi}$$

$$= 3\,\pi a^2/2 \text{ square units.}$$

4. Find the area outside the circle $r = 2\,a\,\cos\theta$ and inside the cardioid $r = a(1 + \cos\theta)$.

Solution. Let r_2 and r_1 represent any corresponding radius vectors of the cardioid and circle respectively. Evidently the element of area has different values in the first and second quadrants. The shaded area in the first quadrant is

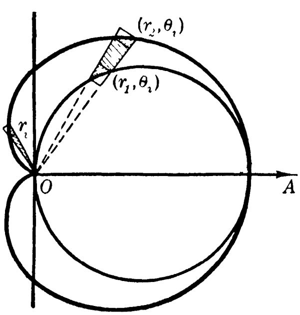


Fig. 170

$$\Delta A_i = \frac{1}{2}(r_2^2 - r_1^2)_i\Delta\theta + e_i\Delta\theta,$$

where $e_i\Delta\theta$ is an infinitesimal of higher order than $(r_2^2 - r_1^2)_i\Delta\theta$.

In the second quadrant the shaded area is

$$\Delta A_i' = \frac{1}{2}(r_2^2)_i \cdot \Delta\theta + \epsilon_i\Delta\theta,$$

and $\epsilon_i\Delta\theta$ is of higher order than $(r_2^2)_i\Delta\theta$.

Therefore

$$A = \lim_{\substack{n \to \infty \\ \Delta\theta \to 0}} \left(\sum_{i=1}^{n} \Delta A_i + \sum_{i=1}^{n} \Delta A_i'\right)$$

$$= a^2 \int_0^{\pi/2} [(1 + \cos\theta)^2 - 4\cos^2\theta]d\theta + a^2 \int_{\pi/2}^{\pi} (1 + \cos\theta)^2 d\theta$$

$$= a^2\left[2\sin\theta - \frac{\theta}{2} - \frac{3}{4}\sin 2\theta\right]_0^{\pi/2} + a^2\left[2\sin\theta + \frac{3\theta}{2} + \frac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi}$$

$$= a^2\left(2 - \frac{\pi}{4}\right) + a^2\left(\frac{3\pi}{4} - 2\right) = \frac{\pi a^2}{2} \text{ square units.}$$

We call attention to the fact that an element like ΔA_i above can only be used if r_2 and r_1 have the same sign for any θ in the interval of integration; otherwise r_2 and r_1 would lie in different quadrants and ΔA_i would not represent an element of area between the curves.

PROBLEMS

Find the area bounded by the following curves. Draw each figure.

1. $x^2 = y + 1$, $x - y + 1 = 0$. *Ans.* $4\frac{1}{2}$ sq. units.

2. $y^2 - x = 10$, $x - 2y + 2 = 0$.

3. $xy = 8$, $2x + y = 10$. *Ans.* $(15 - 16 \log 2)$ sq. units.

4. The parabola $y^2 = 2px$ and its latus rectum.

5. One arch of $y = \cos^2 2x$ and the x axis. *Ans.* $\pi/4$ sq. units.

6. $y = \sin x$ and $y = \cos x$ between two consecutive intersections.

7. $r = a \cos 3\theta$. *Ans.* $\pi a^2/4$ sq. units.

8. $r = a \sin 2\theta$.

9. Smaller loop of $r = 1/2 - \cos\theta$. *Ans.* $(1/8)(2\pi - 3\sqrt{3})$ sq. units.

10. Larger loop of $r = 1/2 + \sin\theta$.

11. Find the area enclosed by $r^2 = a^2 \sin 2\theta$. *Ans.* a^2 sq. units.

12. Prove that the entire area enclosed by the loops of the curve $r = a \sin k\theta$ equals one-fourth, or one-half, the area of the circle of radius a, according as k is an odd or an even integer.

13. Find the area inside $r = 2\sin\theta$ and outside $r = 2(1 - \sin\theta)$.
Ans. $4(\sqrt{3} - \pi/3)$ sq. units.

14. Find the area inside $r = 2$ and outside $r = 4(1 - \cos\theta)$.

15. Find the area of the limaçon $r = 3 + 2\cos\theta$. *Ans.* 11π sq. units.

16. Find the area of the loop of the curve $y^2 = x^2 + x^3$.

17. Find the area within the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.
Ans. $3\pi a^2/8$ sq. units.

18. Find the area of the curve $x^2/a^2 + y^{2/3}/b^{2/3} = 1$.

19. Find the area which is enclosed by the x axis and one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. *Ans.* $3\pi a^2$ sq. units.

20. Find the area under one arch of the curve, $x = a\theta$, $y = a(1 - \cos\theta)$.

21. Find the area between the witch $y = 8a^3/(x^2 + 4a^2)$ and its asymptote, $y = 0$. *Ans.* $4\pi a^2$ sq. units.

22. A goat is tied to a staple on the outer side of a circular fence of radius a units by a rope of length πa units. Find the area over which the goat can graze.

164. Solids of Revolution. The volume of a solid generated by revolving the area bounded by plane curves about some line in the plane may be evaluated very easily by use of the fundamental

theorem. The following examples give very satisfactory elements of volume; one or more such elements can be used in all evaluations of volumes of revolution.

EXAMPLES

1. Find the volume generated by the area under $2y = x^3$ from the origin to the point (2, 4) when revolved about the x axis.

SOLUTION. The shaded area when revolved about $y = 0$ generates

$$\Delta V_i = \pi y_i^2 \cdot \Delta x + k\pi [(y_i + \Delta y_i)^2 - y_i^2]\Delta x, \quad 0 < k < 1,$$

where $\pi k(2y_i \cdot \Delta y_i + \overline{\Delta y_i}^2)\Delta x$, the second term, may be discarded, since it is an infinitesimal of higher order than $\pi y_i^2 \cdot \Delta x$. Therefore we have

$$V = \lim_{\substack{n\to\infty \\ \Delta x \to 0}} \sum_{i=1}^{n} \pi y_i^2 \cdot \Delta x = \pi \int_0^2 y^2\,dx$$

$$= \frac{\pi}{4}\int_0^2 x^6\,dx = \frac{32\pi}{7} \text{ cubic units.}$$

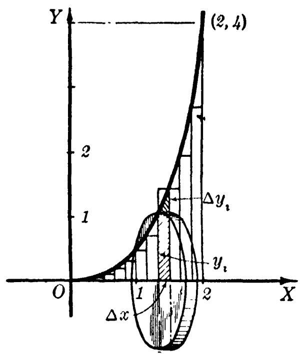


FIG. 171

Since the elements of volume may be considered as $\pi y_i^2 \cdot \Delta x$ without affecting the value of the limit of their sum, we may consider this type of element as a *circular disc* generated by $y_i \cdot \Delta x$ as it turns about the x axis.

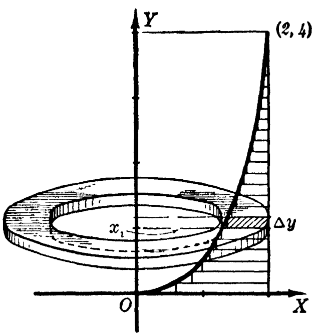


FIG. 172

2. Find the volume generated by revolving the area of Example 1 about the y axis.

SOLUTION. Let the shaded area generate the element of volume; since this element may be considered as the difference of two concentric circular discs, or a ***washer***, we have

$$\Delta V_i = \pi(2)^2\Delta y - \pi x_i^2 \Delta y,$$

discarding infinitesimals of higher order than the first. Therefore

$$V = \lim_{\substack{n\to\infty \\ \Delta y \to 0}} \sum_{i=1}^{n} [\pi(4 - x_i^2)]\Delta y = \pi \int_0^4 (4 - x^2)dy$$

$$= \pi \int_0^4 [4 - (2y)^{2/3}]dy = \frac{32\pi}{5} \text{ cubic units.}$$

In each of these examples the element of volume has been

generated by a rectangle of variable length perpendicular to the axis of revolution. However, the elements of volume assumed different forms. If the rectangle is taken parallel to the axis of revolution, the volume ΔV_i which it generates is a ***thin cylindrical shell.***

Consider the second example when ΔV_i is generated by the shaded area in Fig. 173. Then $\Delta V_i = 2\pi x_i y_i \cdot \Delta x$ except for infinitesimals of order higher than the first. Therefore, applying the fundamental theorem, we get

$$V = \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \sum_{i=1}^{n} \Delta V_i = 2\pi \int_0^2 xy\,dx = \pi \int_0^2 x^4\,dx = \frac{32\pi}{5} \text{ cubic units.}$$

This solution shows that the cylindrical shell element can be considered as the product of the length of the rectangle, the width of the rectangle, and the inner circumference of the shell.

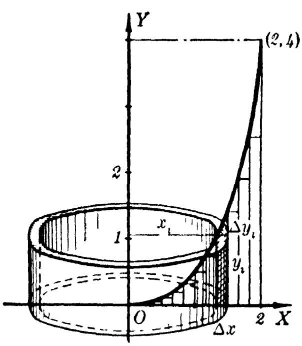


FIG. 173

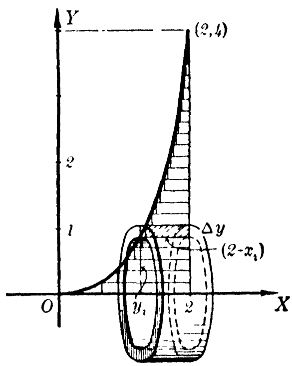


FIG. 174

Figure 174 shows a similar element for Example 1. In this case the fundamental theorem gives for V, in cubic units,

$$V = \lim_{\substack{n \to \infty \\ \Delta y \to 0}} \sum_{i=1}^{n} \Delta V_i = 2\pi \int_0^4 y(2-x)dy = 2\pi \int_0^4 y[2-(2y)^{1/3}]\,dy = \frac{32\pi}{7}.$$

PROBLEMS

1. Find the volume of a sphere as a solid of revolution. Set up the integral using (*a*) disc elements, (*b*) cylindrical shells.

Ans. $4\pi a^3/3$ cubic units.

2. Find the volume of an ellipsoid of revolution by using (*a*) and (*b*) of Problem 1.

3. Find in several ways the volume of a paraboloid of revolution of altitude *h* units and diameter of base 2 *b* units. *Ans.* $\pi hb^2/2$ cubic units.

4. Revolve the area under $y = \log x$ about the x axis and find the volume between $x = 1$ and $x = e$.

Find the volumes generated by the areas bounded by the curves below when revolved about the given line. Use different types of elements. (Nos. 5-17.)

5. $y = x$, $x = 4$, $y = 0$ about $y = -2$. *Ans.* $160\pi/3$ cu. units.

6. $xy = 12$, $x = 2$, $x = 4$, $y = 0$ about $x = -1$.

7. $y = x^2$, $x = 2$, $y = 0$ about $x = 0$. *Ans.* 8π cu. units.

8. $y^2 = 4x$, $x = 4$ about $x = -2$.

9. $x^2 = 4(1 - y)$, $y = 0$ about $y = 3$. *Ans.* $208\pi/15$ cu. units.

10. $xy = 12$, $y = 0$, $x = 2$, $x = 3$ about $y = -7$.

11. $y = x^3$, $x = 1$, $y = 0$ about $x = -2$. *Ans.* $7\pi/5$ cu. units.

12. The loop of $y^2 = x^4(x + 4)$ about the y axis.

13. A circle of radius 3 units about a line in its plane that is 7 units from its center. *Ans.* $126\pi^2$ cu. units.

14. $y = 8/(x^2 + 4)$, $y = 1$, $x = 0$ about $y = 0$; $x = 0$.

15. $y = 2e^{2x}$, $y = e^x$, $x = 0$, $x = 1$ about $y = 1$.
Ans. $(\pi/2)(2e^4 - 5e^2 + 4e - 1)$ cu. units.

16. One arch of $y = \sin x$, $y = 0$, about $y = -2$.

17. One arch of $x = \sin 2y$, $x = 0$, about $x = -2$.
Ans. $\pi(16 + \pi)/4$ cu. units.

18. A vessel has the form generated by revolving $x^2 = 4y$ about $x = 0$. If 4 cu. units of liquid leak out per minute, at what rate is its depth changing?

19. What volume is generated if $r = a(1 - \sin\theta)$ is revolved about its axis?
Ans. $8\pi a^3/3$ cu. units.

20. Prove that the volume formed by rotating the area under $y = \sin x$ from $x = 0$ to $x = \pi$ about the y axis is four times the volume generated when the same area is rotated about the x axis.

165. Length of a Plane Curve. Let s represent the length of a curve $y = f(x)$ from the point $P(a, b)$ to $Q(c, d)$. Divide the interval $a \leqq x \leqq c$ of the x axis into n equal parts Δx_i and erect ordinates to the curve from each point of division. The broken line joining the upper ends of these ordinates has as its length an approximate value of the length of the arc PQ; when the number of segments increases indefinitely and each segment approaches zero as a limit, the length of the broken line approaches the length of the arc PQ as a limit.

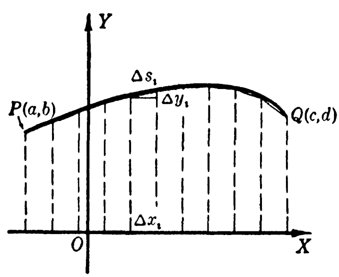


FIG. 175

If each segment of the broken line is represented by Δs_i the length of the broken line is given by

$$\sum_{i=1}^{n} \Delta s_i = \sum_{i=1}^{n} \sqrt{\overline{\Delta x_i}^2 + \overline{\Delta y_i}^2}$$

$$= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i.$$

Now, by the Theorem of Mean Value, § 90, $\Delta y_i/\Delta x_i$ is equal to the value of the derivative dy_i/dx_i for some point (x_i, y_i) of the curve in the interval Δx_i. Therefore

$$s = \lim_{\substack{n \to \infty \\ \Delta s_i \to 0}} \sum_{i=1}^{n} \Delta s_i = \lim_{\substack{n \to \infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{dy_i}{dx_i}\right)^2} \Delta x_i,$$

or

(1) $$s = \int_a^c \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

if no ordinate of the curve meets the arc more than once.*

If y is used as the variable of integration, the same process, when Δy_i is factored from the radical, gives

(2) $$s = \int_b^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy,$$

if no abscissa meets the arc of the curve more than once.*

The integral giving the length of a curve may be written

(3) $$s = \int \sqrt{dx^2 + dy^2}.$$

Then if x and y are each expressed in terms of a third variable, dx and dy can be replaced by their values in terms of that variable.

In polar coordinates, by the substitution of their values for dx and dy in (3), we have

(4) $$s = \int \sqrt{dr^2 + r^2 d\theta^2},$$

and we choose as the variable of integration the one which makes the integration simpler. In using (4), the dependent variable must be a single-valued function of the variable of integration.*

* If this condition is not satisfied, consider smaller pieces of the arc for each of which the condition holds.

EXAMPLES

1. Find the length of the curve $9x^2 = 4(1 + y^2)^3$ from $(2/3, 0)$ to the point $(10\sqrt{5}/3, 2)$.

SOLUTION. It appears easier to solve for x than for y. We have therefore

$$x = \pm \frac{2}{3}(1 + y^2)^{3/2}$$

whence

$$\frac{dx}{dy} = 2y(1 + y^2)^{1/2},$$

if $x > 0$. Then

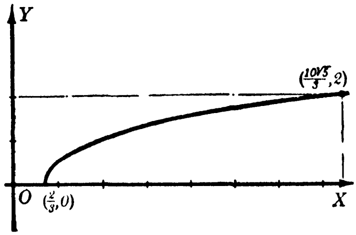


FIG. 176

$$s = \int_0^2 \sqrt{1 + 4y^2(1 + y^2)}\, dy = \int_0^2 (1 + 2y^2)dy$$

$$= \left(y + \frac{2y^3}{3}\right)\Big]_0^2 = 7\frac{1}{3} \text{ units.}$$

2. Find the length of one-half of one arch of the cycloid $x = 2(\theta - \sin\theta)$, $y = 2(1 - \cos\theta)$.

SOLUTION. Since the curve is given by parametric equations, we shall use the differential form (3) of the formula for the length of a curve and simplify after substituting for dx and dy. That is,

$$s = \int_{\theta=0}^{\theta=\pi} \sqrt{dx^2 + dy^2} = \int_0^\pi \sqrt{4(1 - \cos\theta)^2 + 4\sin^2\theta}\, d\theta$$

$$= 2\int_0^\pi \sqrt{2 - 2\cos\theta}\, d\theta = 4\int_0^\pi \sin\frac{\theta}{2}\, d\theta.$$

$$= 4\left[-2\cos\frac{\theta}{2}\right]_0^\pi = 8 \text{ units.}$$

3. Find the length of the curve $r = a(1 - \sin\theta)$.

SOLUTION. Using the form $\sqrt{dr^2 + r^2d\theta^2}$, we have, from symmetry,

$$s = 2\int_{\pi/2}^{3\pi/2} \sqrt{a^2\cos^2\theta + a^2(1 - \sin\theta)^2}\, d\theta$$

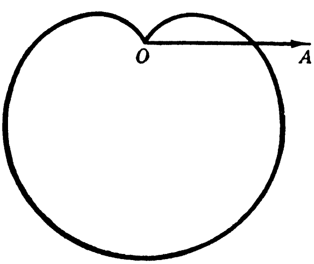


FIG. 177

$$= 2a\int_{\pi/2}^{3\pi/2} \sqrt{2 - 2\sin\theta}\, d\theta$$

$$= 2a\sqrt{2}\int_{\pi/2}^{3\pi/2} \sqrt{1 - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\, d\theta$$

$$= 2a\sqrt{2}\int_{\pi/2}^{3\pi/2} \left[\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right] d\theta$$

$$= 2a\sqrt{2}\left[-2\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right]_{\pi/2}^{3\pi/2}$$

$$= 8a \text{ units.}$$

PROBLEMS

1. A curve has $ds = (1 + x^2)dx$. What is its length from $x = 1$ to $x = 3$? *Ans.* $10\frac{2}{3}$ units.

Find the length of each of the following curves. (Nos. 2–21.)

2. $9y^2 = 4(1 + x^2)^{2/3}$ from $x = 0$ to $x = 3$.

3. The semi-cubical parabola $y^2 = (x - 2)^3$ from $(2, 0)$ to $(6, 8)$. *Ans.* 9.07 units.

4. The semi-cubical parabola $y^3 = x^2$ in the first quadrant from $y = 0$ to $y = 4$.

5. $y = \log \sec x$ from $(0, 0)$ to $(\pi/3, \log 2)$. *Ans.* $\log \tan (5\pi/12)$ units $= \log (2 + \sqrt{3}) = 1.317$ units.

6. $y = a \log x$ from $(1, 0)$ to $(a, a \log a)$.

7. $y = \log (1 - x^2)$ from $x = 0$ to $x = 1/2$. *Ans.* $\log 3 - 1/2$ units.

8. $y = \log \csc x$ from $x = \pi/3$ to $x = \pi/2$.

9. $x^{2/3} + y^{2/3} = a^{2/3}$. *Ans.* $6a$ units.

10. The hypocycloid $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

11. One arch of the *cycloid* $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$. *Ans.* $8a$ units.

12. $y = 2 \log [4/(4 - x^2)]$ from $x = 0$ to $x = 1$.

13. $y = 4 \log [16/(16 - x^2)]$ from $x = 0$ to $x = 2$. *Ans.* $4 \log 3 - 2$ units.

14. The cardioid $r = 2(1 - \sin \theta)$.

15. $r = a \cos^4(\theta/4)$ from $\theta = 0$ to $\theta = 2\pi$. *Ans.* $8a/3$ units.

16. The cardioid $r = a(1 + \cos \theta)$.

17. $r = a \cos^3(\theta/3)$ from $\theta = 0$ to $\theta = 3\pi$. *Ans.* $3\pi a/2$ units.

18. $x = e^{-t} \cos 2t$, $y = e^{-t} \sin 2t$ from $t = 0$, to $t = \pi/4$.

19. $x = e^{-(3/2)t} \cos t$, $y = e^{-(3/2)t} \sin t$, from $t = 2a/3$ to $t = a$. *Ans.* $(\sqrt{13}/3)(e^{-a} - e^{-(3/2)a})$ units.

20. The involute $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$ from $t = 0$ to $t = t_1$.

21. The *catenary* $y = (a/2)(e^{x/a} + e^{-x/a})$ from $x = 0$ to $x = x_1$. *Ans.* $(a/2)(e^{x_1/a} - e^{-x_1/a})$ units.

166. Mean Value of a Function. The limit of the average value of a finite number of values of a function of x, taken at equal intervals of x, as their number increases without limit is called the ***mean value*** of the function.

Let $q(x_i)$, $i = 0, 1, 2, \cdots, n-1$, represent n values of the function $q(x)$ such that $n\cdot\Delta x = k$, the interval of x. Then the ***average value*** of the $q(x_i)$ is

$$\frac{q(x_0) + q(x_1) + q(x_2) + \cdots + q(x_{n-1})}{n}.$$

Using the summation symbols and replacing n by its value $k/\Delta x$, we may write this in the form

$$\text{Average of } q(x_i) = \frac{1}{k}\sum_{i=0}^{n-1} q(x_i)\cdot\Delta x.$$

Hence, from the definition above,

$$\text{Mean value of } q(x) = \frac{1}{k}\lim_{\substack{n\to\infty\\ \Delta x\to 0}}\sum_{i=0}^{n-1} q(x_i)\cdot\Delta x = \frac{1}{k}\int_a^b q(x)dx,$$

where a and b are derived from the interval over which x is allowed to vary in computing the average value of $q(x_i)$.

EXAMPLES

1. Find the mean value of the ordinate to the curve $y = \cos x$ if the ordinates are erected at equal intervals from $x = 0$ to $x = \pi/2$.

SOLUTION. Assume n such ordinates in the given interval. Then calling the average ordinate A, we have by definition

$$A = \frac{y_0 + y_1 + y_2 + \cdots + y_{n-1}}{n}$$

$$= \frac{2}{\pi}\sum_{i=0}^{n-1} y_i\cdot\Delta x, \text{ since } n\cdot\Delta x = \frac{\pi}{2}.$$

Hence

$$\text{Mean ordinate} = \frac{2}{\pi}\lim_{\substack{n\to\infty\\ \Delta x\to 0}}\sum_{i=0}^{n-1} y_i\cdot\Delta x$$

$$= \frac{2}{\pi}\int_0^{\pi/2} y\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\cos x\,dx = \frac{2}{\pi}\text{ units.}$$

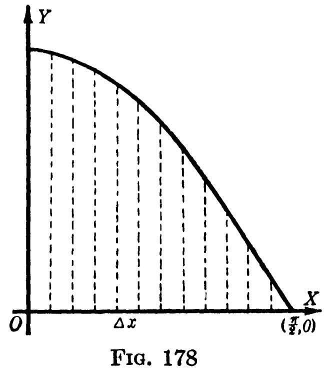


FIG. 178

2. Find the mean value of the area of squares constructed on a set of parallel chords of a circle if the chords are at equal intervals along the circumference of the circle.

SOLUTION. Suppose the square $ABCD$ constructed on AB, any one of the

chords. Such a chord is $2x_i$ in length and hence each square has the area $4x_i^2$. The equal intervals are Δs_i and $n \cdot \Delta s_i = \pi r$. Therefore

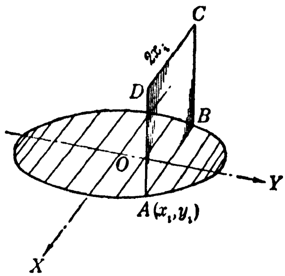


FIG. 179

$$\text{Average area} = \frac{4x_0^2 + 4x_1^2 + \cdots + 4x_{n-1}^2}{n}$$

$$= \frac{4}{\pi r}\sum_{i=0}^{n-1} x_i^2 \cdot \Delta s_i.$$

Hence

$$\text{Mean area} = \frac{4}{\pi r}\lim_{\substack{n \to \infty \\ \Delta s_i \to 0}}\sum_{i=0}^{n-1} x_i^2 \cdot \Delta s_i$$

$$= \frac{4}{\pi r}\int_{y=-r}^{y=r} x^2\, ds$$

$$= \frac{4}{\pi}\int_{-r}^{r} \sqrt{r^2 - y^2}\, dy,$$

since ds for the circle is equal to $(r/x)dy$. Accordingly, we have

$$\text{Mean area} = \frac{4}{\pi}\left[\frac{1}{2}\left(y\sqrt{r^2 - y^2} + r^2 \sin^{-1}\frac{y}{r}\right)\right]_{-r}^{r}$$

$$= 2r^2 \text{ square units.}$$

PROBLEMS

1. Find the mean vertical width of the area bounded by $y = x^2$ and $x = y^2$ with respect to, or at, equal intervals of x. *Ans.* 1/3 unit.

2. Find the mean vertical width of the loop of $y^2 = x^2(4 - x)$ if measured at equal intervals along the x axis.

3. Lines are drawn parallel to the x axis in the area bounded by $xy = 6$, $y = x$, $y = 2$. If these lines are equally spaced along the y axis, what is their mean length? *Ans.* $[3 \log (3/2) - 1]/(\sqrt{6} - 2)$ units.

4. What is the mean value of $x \log x$ with respect to x from $x = 2$ to $x = 8$?

5. The same as Problem 4 for $x(\log x)^2$ from $x = 1$ to $x = 4$.
Ans. $(1/12)[32 (\log 4)(\log 4 - 1) + 15]$ units.

6. What is the mean value of the square of the ordinate of $y = +\sqrt{4 - x^2}$ with respect to x?

7. What is the mean length of the radius vector of $r = 4 \sin 2\theta$ for one loop with respect to θ? *Ans.* $8/\pi$ units.

8. The same as Problem 7 for $r = a(1 + \cos \theta)$ in the first quadrant.

9. The same as Problem 7 for $r^2 = a^2(1 + \cos \theta)$ in the first quadrant.

10. Find the mean area of isosceles triangles inscribed in a circle of radius 10 units if they have a common vertex and successive bases cut off equal segments on the diameter through that vertex.

11. The same as Problem 10 for equal arcs cut off by successive bases.
Ans. $200/\pi$ sq. units.

12. What is the mean volume of cylinders inscribed in a sphere if their altitudes have a common difference?

13. The same as Problem 12 for diameters with a common difference.
Ans. $\pi^2a^3/8$ cu. units.

14. Find the mean area of circles on the double ordinates of $4x^2 + 9y^2 = 36$ as diameters, if they are spaced equally along the x axis.

15. What is the mean area of right triangles in $r = a\cos\theta$, with the hypotenuses along the initial line and radius vectors at equal intervals of θ from 0 to $\pi/2$ as one side? *Ans.* $a^2/2\pi$ sq. units.

16. Find the mean value of the widths of the parabolic segment $y^2 = 2x$, $x = 3$, (a) if the widths are taken at equal intervals along the y axis; (b) the x axis. Why are the results different?

17. Rectangles are inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$. If the altitudes are at equal intervals along the x axis, prove that the mean value of the area is the same as when the bases are at equal intervals along the y axis.

18. Find the mean ordinate of a quadrant of a circle if the ordinates are at equal intervals along (a) the arc; (b) the x axis; (c) the y axis.

ADDITIONAL PROBLEMS

Find the areas bounded as follows. (Nos. 1–9.)

1. $r = a\sin 4\theta$. *Ans.* $\pi a^2/2$ sq. units.

2. $y = x^2$, $y = 8 - x^2$.

3. $r = a(1 + \cos\theta)$. *Ans.* $3\pi a^2/2$ sq. units.

4. $4x^2 - 9y^2 + 36 = 0$, above $y = 1$.

5. $x^2/9 + y^2/16 = 1$. *Ans.* 12π sq. units.

6. $r^2 = 9\sin 2\theta$.

7. Outside $r = a$, and inside $r = 2a\sin 2\theta$.
Ans. $a^2(\sqrt{3} + 2\pi/3)$ sq. units.

8. Outside $r = a\sin\theta$, and inside $r = a(3 + 2\sin\theta)$.

9. Inside $r = 5\sin\theta$ and outside $r = 2 + \sin\theta$.
Ans. $\sqrt{3} + 8\pi/3$ sq. units.

Find the volume of the solid of revolution generated by rotating each of the areas defined below about the line specified. (Nos. 10–15.)

10. $y^2 = 8x$, $x = 2$ about $x = -2$.

11. $y^2 = x$, $x + y = 2$ about the y axis. *Ans.* $(14\frac{2}{5})\ \pi$ cu. units.

12. $16\,x = y^2$, $4\,x = y^2 - 12$ about $x = 1$.

13. $y = x^2 + 1$, $y = x + 3$ about $x = 2$. *Ans.* $(13\frac{1}{2})\ \pi$ cu. units.

14. $y = e^{-x}$ from $x = 0$ to $x = \infty$ about $y = 0$.

15. A parabolic segment of altitude a units and base $2\,b$ units about its base. *Ans.* $16\,\pi a^2 b/15$ cu. units.

Find the length of each of the following curves. (Nos. 16–17.)

16. $x = t$, $y = \log \sec t$ from $t = 0$ to $t = \pi/4$.

17. $y = x^3/6 + 1/(2x)$ from $x = 1$ to $x = 3$. *Ans.* $4\frac{2}{3}$ units.

18. Find the perimeter of the area bounded by $y = e^x + e^{-x}$ and $y = 2$.

19. A suspended wire is in the form of a parabola. Its ends are $2\,l$ units apart on a horizontal line and its dip is h units. Find its length.
Ans. $\sqrt{l^2 + 4\,h^2} + (l^2/2\,h) \log [(2\,h + \sqrt{l^2 + 4\,h^2})/l]$ units.

20. Find the mean vertical width of the loop of $4\,y^2 = x^2(4 - 2\,x)$ with respect to x.

21. Find the mean value of the square of the ordinates of the semicircle $y = +\sqrt{9 - x^2}$ if they are equally spaced along the x axis. *Ans.* 6 sq. units.

22. Problem 21 except the ordinates are equally spaced along the arc.

23. Find the mean velocity of a falling body with respect to the distance fallen, if it falls from rest for 5 seconds. *Ans.* $(3\frac{1}{3})g$ ft./sec.

24. Find the mean volume of cylinders inscribed in a right circular cone of altitude 8 in. and radius 4 in., if the upper bases are equally spaced along the axis of the cone.

25. Find the mean volume of cylinders inscribed in a sphere if the angles at the center subtended by the altitudes have a common difference.
Ans. $4\,a^3/3$ cu. units.

26. Find the volume common to two equal circular cylinders if their axes meet at right angles.

27. When a liquid flows through a straight pipe of radius a, the velocity of flow at a distance r from the center of the pipe is $k(a^2 - r^2)$. Find the mean value of the velocity along a diameter. *Ans.* $2\,ka^2/3$ units per unit of time.

28. Oil is poured into a tank containing water so that the density of the mixture is decreasing at the rate of 0.2 lb. per minute. Find the mean value of the density of the mixture with respect to the time during 5 minutes.
Ans. 62 lbs.

29. Assuming that oil flowing in a straight pipe of inside radius a moves in straight lines parallel to the axis of the pipe, and that the velocity at a distance r from the axis is $v_0(1 - r^2/a^2)$, where v_0 is the velocity at the center, find what volume of oil will flow past a given point per second.

30. A carpenter chisels a square hole of side r inches through a round post of radius r inches; the axis of the hole meets that of the post at right angles and two sides of the hole are parallel to the axis of the post. How much wood is cut away?

31. Two equal circular cylinders of radius 3 inches intersect so that their axes meet at an angle of 60°. Find the volume of the part common to the two cylinders. *Ans.* $96\sqrt{3}$ cu. in.

32. Given the curve $y = (\log x)/\sqrt{x}$. Find the volume of the solid of revolution formed by revolving about the x axis the area bounded by the curve, the x axis, the ordinate through the maximum point, and that through the inflection of the curve.

33. Find the mean horizontal width of the area bounded by $y = x^{1/2}$ and $y = x^3$ with respect to y. *Ans.* 5/12 unit.

34. Problem 33 with respect to x along the first curve.

35. Problem 33 with respect to x along the second curve. *Ans.* 5/14 unit.

CHAPTER XIV

MULTIPLE INTEGRATION

167. Areas by Double Integration. If the bounding curves of a plane area are given in rectangular coordinates, we may divide the area into small elements of dimensions Δx by Δy by means of two sets of parallel lines at intervals of Δx and Δy along the x axis and the y axis, respectively. The sum of the areas of those elements which lie entirely within the boundary may be used as an approximation of the enclosed area. The limit of such a sum as both Δx and Δy approach zero as a limit is the area enclosed by the curves. That is,

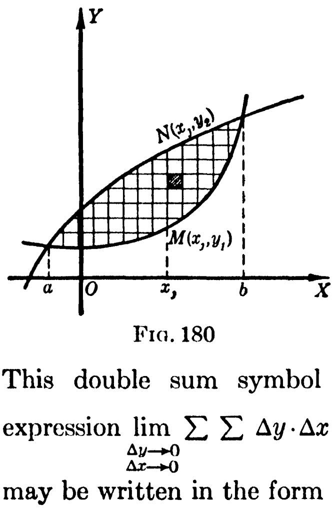


FIG. 180

$$A = \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \sum \sum \Delta y \cdot \Delta x.$$

This double sum symbol means $\sum (\sum \Delta y) \Delta x$. Therefore the expression $\lim\limits_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \sum \sum \Delta y \cdot \Delta x$ means $\lim\limits_{\substack{m \to \infty \\ \Delta x_j \to 0}} \sum\limits_{j=1}^{m} (\lim\limits_{\substack{n \to \infty \\ \Delta y_i \to 0}} \sum\limits_{i=1}^{n} \Delta y_i) \Delta x_j$, which may be written in the form

$$\lim_{\substack{m \to \infty \\ \Delta x_j \to 0}} \sum_{j=1}^{m} \left(\int_{y_1}^{y_2} dy \right) \Delta x_j,$$

where the fundamental theorem has been applied to the quantity in the parenthesis. During this process x_j and Δx_j are kept constant. Now, considering the parenthesis of the last expression as the $f(x_j)$ of the fundamental theorem, since y_1 and y_2 are functions of x_j alone, we have, by a second application of the theorem,

$$(1) \qquad A = \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \sum \sum \Delta y \cdot \Delta x = \int_a^b \left(\int_{y_1}^{y_2} dy \right) dx.$$

This is called a ***double integral;*** it gives the value of A and is usually written in the form

$$A = \int_a^b \int_{y_1}^{y_2} dy\,dx.$$

It should be observed that the *second integral sign belongs with the first differential in such double integrals.* Also it must be noted that during the first integration, which gives the area of the strip MN, x does not vary. The limits on the second integral sign are *usually variables* defined by the $y = f_i(x)$ equations representing the curves bounding the area, but the limits on the first integral sign are *always constants,* the extreme values of the last variable of integration for the area being evaluated.

If the fundamental theorem is applied to the double sum in reverse order, we get

$$(2) \qquad A = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum \sum \Delta x \cdot \Delta y = \int_c^d \int_{x_1}^{x_2} dx\,dy.$$

Reversing the order of integration is sometimes very desirable, as it may simplify a difficult integral.

Performing the first integration of (1), we have

$$A = \int_a^b (y_2 - y_1)dx,$$

which is the form of the single integral representing the desired area. This fact should help the student to write the proper limits for the double integral.

If the plane area is bounded by curves given in polar coordinates, it may also be evaluated by double integration.

Let the area OPQ be divided into small four-sided elements by means of concentric circles and radial lines at intervals of Δr and $\Delta\theta$ respectively. The element shaded in Fig. 181 is

$$\Delta A_{ij} = \frac{1}{2}(r_i + \Delta r)^2 \Delta\theta - \frac{1}{2} r_i{}^2 \Delta\theta$$

$$= r_i \cdot \Delta r \cdot \Delta\theta + (1/2)\,\overline{\Delta r}^2 \cdot \Delta\theta.$$

The quantity $(1/2)\overline{\Delta r}^2 \cdot \Delta\theta$ is an infinitesimal of higher order than

$r_i \cdot \Delta r \cdot \Delta\theta$, as it can be written $(1/2)(\Delta r)(\Delta r \cdot \Delta\theta)$; hence it can be omitted from the element. Then we have

$$(3) \qquad A = \lim_{\substack{m,\, n \to \infty \\ \Delta r \to 0 \\ \Delta\theta \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} r_i \cdot \Delta r \cdot \Delta\theta$$

$$= \int_{\theta_1}^{\theta_2} \int_0^{f(\theta)} r\, dr\, d\theta,$$

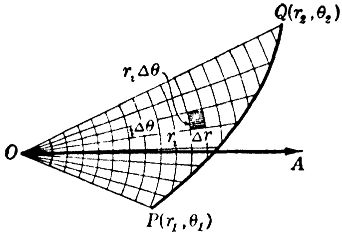


FIG. 181

where $r = f(\theta)$ is the equation of the curve PQ.

We shall illustrate the evaluation of areas by double integration by solving examples previously solved by single integration in § 163. The student should compare the two methods and observe that in the new solution of each example the final integral obtained is the integral used in the previous solution.

EXAMPLES

1. Find the area bounded by $y = 0$ and $y = 4x - x^2$.

SOLUTION. Suppose the desired area divided into elements Δy by Δx as shown in Fig. 182. Then $\Delta A = \Delta y \cdot \Delta x$, and

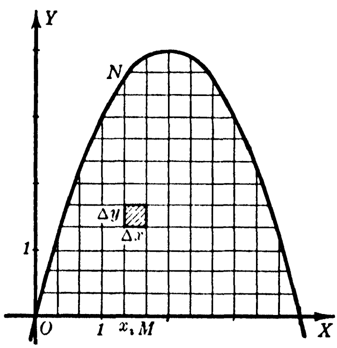


FIG. 182

$$A = \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \Sigma\Sigma \Delta y \cdot \Delta x$$

$$= \int_0^4 \int_0^{4x - x^2} dy\, dx,$$

since $y_1 = 0$ and $y_2 = 4x - x^2$ and the extreme limits of x are 0 and 4.

Integrating with respect to y, we have

$$A = \int_0^4 y \Big]_0^{4x - x^2} dx$$

$$= \int_0^4 (4x - x^2)dx,$$

where the integrand represents the strip which was used as the element in single integration. The final integration gives

$$A = \left(2x^2 - \frac{x^3}{3}\right)\Big]_0^4 = \frac{32}{3} \text{ square units.}$$

2. Find the area bounded by the two curves $y = x^3 - x^2$ and $y = x^2$.

SOLUTION. The limits for x are found to be 0 and 2 by solving the equations of the curves simultaneously.

Hence, since $\Delta A = \Delta y \cdot \Delta x$,

$$A = \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \Sigma\Sigma \Delta y \cdot \Delta x$$

$$= \int_0^2 \int_{x^3 - x^2}^{x^2} dy\, dx.$$

Whence

$$A = \int_0^2 y \Big]_{x^3 - x^2}^{x^2} dx$$

$$= \int_0^2 (2\,x^2 - x^3)dx = \frac{4}{3} \text{ square units.}$$

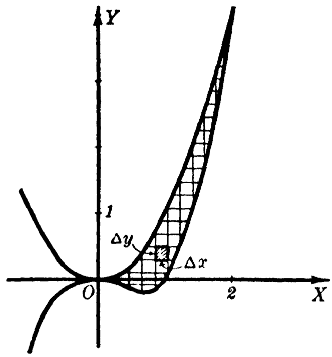


FIG. 183

3. Find the area enclosed by the cardioid $r = a(1 + \cos\theta)$.

SOLUTION. Here we may take as the element of area ΔA_{ij} the infinitesimal $r_i \cdot \Delta r \cdot \Delta\theta$ as explained above. Hence

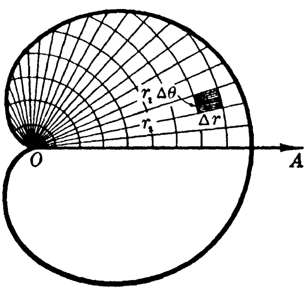


FIG. 184

$$A = \lim_{\substack{m,\, n \to \infty \\ \Delta r \to 0 \\ \Delta\theta \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} r_i \cdot \Delta r \cdot \Delta\theta$$

$$= 2 \int_0^{\pi} \int_0^{a(1+\cos\theta)} r\, dr\, d\theta,$$

The first integration with respect to r gives

$$A = 2 \int_0^{\pi} \left(\frac{r^2}{2}\right)\Big]_0^{a(1+\cos\theta)} d\theta$$

$$= a^2 \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)d\theta.$$

Integrating this, we get, as previously,

$$A = \frac{3\,\pi a^2}{2} \text{ square units.}$$

4. Find the area outside the circle $r = 2\,a\cos\theta$ and inside the cardioid $r = a(1 + \cos\theta)$.

SOLUTION. With ΔA_{ij} represented by $r_i \cdot \Delta r \cdot \Delta\theta$, the area desired in the first and fourth quadrants is given by

$$\lim_{\substack{m,\, n \to \infty \\ \Delta r \to 0 \\ \Delta\theta \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} r_i \cdot \Delta r \cdot \Delta\theta = 2\int_0^{\pi/2} \int_{2\,a\cos\theta}^{a(1+\cos\theta)} r\, dr\, d\theta,$$

and that in the second and third quadrants is

$$\lim_{\substack{m,\, n \to \infty \\ \Delta r \to 0 \\ \Delta\theta \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} r_i \cdot \Delta r \cdot \Delta\theta = 2 \int_{\pi/2}^{\pi} \int_0^{a(1+\cos\theta)} r\, dr\, d\theta.$$

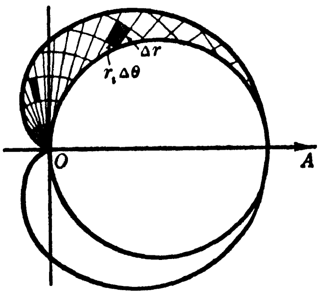


FIG. 185

Therefore the total area is

$$A = 2 \int_0^{\pi/2} \int_{2a\cos\theta}^{a(1+\cos\theta)} r\, dr\, d\theta + 2 \int_{\pi/2}^{\pi} \int_0^{a(1+\cos\theta)} r\, dr\, d\theta,$$

and integration with respect to r gives

$$A = a^2 \int_0^{\pi/2} (1 + 2\cos\theta - 3\cos^2\theta)d\theta + a^2 \int_{\pi/2}^{\pi} (1 + 2\cos\theta + \cos^2\theta)d\theta.$$

These integrals with the combined value $\pi a^2/2$ square units appeared in the solution of this example by single integration.

PROBLEMS

Find by the method of double integration each of the areas bounded as follows. (Nos. 1–5.)

1. The semi-cubical parabola $y^2 = x^3$ and the line $y = 2x$.
Ans. 16/5 sq. units.

2. Inside the circle $x^2 - 4ay + y^2 = 0$ and outside the parabola $x^2 = 2ay$.

3. The segment of $r = 8$ and $r\cos\theta = 4$ which does not include the pole.
Ans. $16(4\pi/3 - \sqrt{3})$ sq. units.

4. Inside $r = 2a(1 - \cos\theta)$ and outside $r = a$.

5. Between the parabola $x^2 = 2ay$ and the witch $y = a^3/(x^2 + a^2)$.
Ans. $a^2(\pi/2 - 1/3)$ sq. units.

6. In Problems 1 and 2, reverse the order of integration, change the limits and check the results.

Find each of the following areas by double integration. (Nos. 7–19.)

7. Between $y = xe^{-x}$ and the x axis. What part of this area lies between the high point and the point of inflection?
Ans. 1 sq. unit; $(2/e - 3/e^2)$ sq. units.

8. The loop of $2y^2 = x^2(2 - x)$.

9. Between the circles $r = a \sin \theta$ and $r = a \cos \theta$.
Ans. $a^2(\pi - 2)/8$ sq. units.

10. Outside the circle $r = 3a/2$ and inside the cardioid $r = a(1 + \cos \theta)$.

11. Enclosed by $x^2 y = \log x$, $y = 0$ and $x = e^{1/2}$.
Ans. $(1 - 1/\sqrt{e})$ sq. units.

12. Under one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

13. Enclosed by the hypocycloid $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.
Ans. $3\pi a^2/8$ sq. units.

14. Between $x^{1/2} + y^{1/2} = a^{1/2}$ and $x + y = a$.

15. Enclosed by the lemniscate $r^2 = a^2 \cos 2\theta$. *Ans.* a^2 sq. units.

16. Enclosed by the three-leafed rose $r = a \sin 3\theta$.

17. Enclosed by the hyperbolic spiral $r\theta = 2c$ and any two vectors r_1 and r_2 drawn to it. *Ans.* $c(r_1 - r_2)$ sq. units.

18. Between the axes, the catenary $y = (a/2)(e^{x/a} + e^{-x/a})$, and $x = b$.

19. The area formed by the x axis and any arch of $y = \sin x$ is divided into two parts by the curve $y = \cos x$. Find the area of each part.
Ans. $2 - \sqrt{2}$, and $2 + \sqrt{2}$ sq. units.

20. P_1 and P_2 are any two points on the hyperbola $xy = k$. Prove that the area between the arc P_1P_2 and the x axis is equal to the area between the same arc and the y axis.

168. Solids of Revolution by Double Integration. If the area which generates the volume of a solid of revolution is divided into elements Δy by Δx and the element in the jth row and the ith column is revolved about the line $y = c$, the volume generated is

$$\Delta V_{ij} = \pi(r_{ij} + \Delta y)^2 \cdot \Delta x - \pi r_{ij}^2 \cdot \Delta x,$$
$$= 2\pi r_{ij} \cdot \Delta y \cdot \Delta x + \pi \cdot \overline{\Delta y}^2 \cdot \Delta x,$$

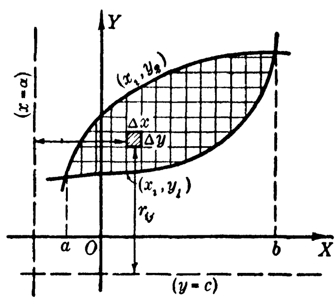


FIG. 186

where $\pi \cdot \overline{\Delta y}^2 \cdot \Delta x$ is of higher order than $2\pi r_{ij} \cdot \Delta y \cdot \Delta x$. Hence by the fundamental theorem we have

$$V = \lim_{\substack{m, n \to \infty \\ \Delta y \to 0 \\ \Delta x \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} 2\pi r_{ij} \cdot \Delta y \cdot \Delta x = 2\pi \int_a^b \int_{y_1}^{y_2} r \, dy \, dx.$$

Similarly for polar coordinates, if an area is revolved about the initial line the circular ring generated is $2\,\pi r_i{}^2 \sin\theta_j \cdot \Delta r \cdot \Delta\theta$ except for infinitesimals of higher order and therefore

$$V = \lim_{\substack{m,\,n\to\infty \\ \Delta r\to 0 \\ \Delta\theta\to 0}} \sum_{j=1}^{m}\sum_{i=1}^{n} 2\,\pi r_i{}^2 \sin\theta_j \cdot \Delta r \cdot \Delta\theta = 2\,\pi \int_\alpha^\beta \int_{r_1}^{r_2} r^2 \sin\theta \, dr\, d\theta.$$

In this element $\sin\theta$ becomes $\cos\theta$ if the volume has the line $\theta = \pi/2$ as its axis.

EXAMPLE

Solve the first example given in § 164 on volumes of revolution.

SOLUTION. In this example $r_{ij} = y_i$, so

$$\Delta V_{ij} = \pi(y_i + \Delta y)^2 \cdot \Delta x - \pi y_i{}^2 \cdot \Delta x$$
$$= 2\,\pi y_i \cdot \Delta y \cdot \Delta x + e_i \cdot \Delta x.$$

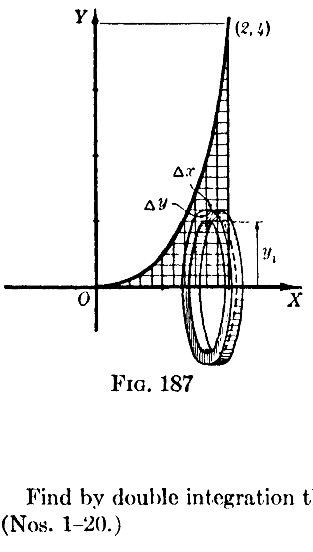


FIG. 187

Therefore

$$V = \lim_{\substack{m,\,n\to\infty \\ \Delta y\to 0 \\ \Delta x\to 0}} \sum_{j=1}^{m}\sum_{i=1}^{n} 2\,\pi y_i \cdot \Delta y \cdot \Delta x$$
$$= 2\,\pi \int_0^2 \int_0^{x^3/2} y\, dy\, dx$$
$$= 2\,\pi \int_0^2 \left(\frac{y^2}{2}\right)\Big]_0^{x^3/2} dx$$
$$= \frac{\pi}{4}\int_0^2 x^6\, dx = \frac{32\,\pi}{7} \text{ cubic units.}$$

PROBLEMS

Find by double integration the volumes of each of the solids defined below. (Nos. 1–20.)

1. A sphere by rectangular coordinates; by polar coordinates.
Ans. $4\,\pi a^3/3$ cu. units.

2. Area between $x^2 = 4\,y$ and $y = 4$, revolved about $y + 4 = 0$.

3. Area between $4\,x = 16\,y - y^2$ and $x = 0$, revolved about $y = 0$.
Ans. $(2730\tfrac{2}{3})\,\pi$ cu. units.

4. Area between $y = x^2$, $x = 2$, and $y = 1$, revolved about $x = 0$.

5. Area between $16\,y^2 = (x + 4)^3$ and its tangent at (12, 16), revolved about $y = 0$.
Ans. $(113\tfrac{7}{9})\,\pi$ cu. units.

6. Area between $x^2 - 8\,y + 23 = 0$ and $x^2 + y - 13 = 0$, revolved about $x = 0$.

7. Area bounded by $y^2 = x(x-1)(x-2)$ from $x = 2$ to $x = 3$, revolved about $y = 0$. *Ans.* $9\pi/4$ cu. units.

8. Area between $y^2 = 4x + 4$ and $x = 3$, revolved about $x + 4 = 0$.

9. Area between $x = (y+1)^2$, $x = 1$, and $y = 1$, revolved about $y = 0$. *Ans.* $11\pi/6$ cu. units.

10. Area between $5y^2 = 80 - 16x$ and $x = 0$, revolved about $x + 1 = 0$.

11. Area in a parabolic segment 8 units high, with a base of 10 units, revolved about the base. *Ans.* $(341\frac{1}{3})\pi$ cu. units.

12. Area of a circle of radius 2 units, revolved about a line 3 units from the center.

13. Area between one arch of $y = \sin 3x$ and $x = 0$, revolved about $y + 1 = 0$. *Ans.* $\pi(8+\pi)/6$ cu. units.

14. Area in the first quadrant bounded by $y(x^2+4) = 8$, $x = 0$, and $y = 1$, revolved about the y axis.

15. Area bounded by $y(2x-1)^2 = 4$, $x = 0$, $x = 1/2$ and $y = 0$, revolved about $y = 0$. *Ans.* Not finite.

16. A football, if a section containing a seam is an ellipse 12 in. by 8 in.

17. Area bounded by $y = 2e^{2x}$, $y = e^x$, $x = 0$, and $x = 1$, revolved about $y = 1$. *Ans.* $(\pi/2)(2e^4 - 5e^2 + 4e - 1)$ cu. units.

18. Area bounded by a cardioid, revolved about its axis.

19. Area bounded by $r = a\cos\theta$, revolved about $\theta = \pi/2$. *Ans.* $\pi^2a^3/4$ cu. units.

20. Area bounded by the first-quadrant loop of $r = 2\sin 2\theta$, revolved about $\theta = \pi/2$.

21. The arc of the curve $xy = x - y$ joining the origin to any point $P_1(x_1, y_1)$ of the curve bounds with the x axis and the line $x = x_1$ an area A. The same arc bounds with the y axis and the line $y = y_1$ an area B. Show that the volumes generated when A is revolved about $y = 0$ and B about $x = 0$ are equal for any point P_1.

22. Derive the general double integration element for polar coordinates if an area bounded by $r_1 = \phi_1(\theta)$ and $r_2 = \phi_2(\theta)$ is revolved about a line parallel to the polar axis; perpendicular to the polar axis.

169. Non-symmetric Volumes by Double Integration. Just as we have the problem of finding the area bounded by given curves, we also have that of finding the volume bounded by given surfaces.

Suppose the surface S of Fig. 188 is projected upon the xy plane and the volume under this surface bounded below by the plane and laterally by the cylinder of projection is desired. Divide the base of the required solid into elements Δy by Δx. The volume of the column over the element of area in the ith row and the jth strip may be considered as an element of volume ΔV_{ij}. Then

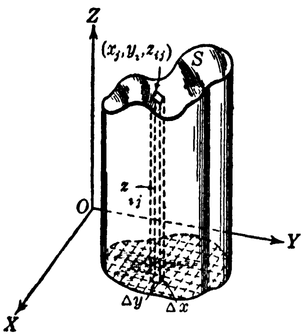


FIG. 188

$$\Delta V_{ij} = z_{ij} \cdot \Delta y \cdot \Delta x + k \cdot \Delta z_{ij} \cdot \Delta y \cdot \Delta x,$$

where $|k| < 1$, and where $z = f(x, y)$ is the equation of the surface S and $k\Delta z_{ij} \cdot \Delta y \cdot \Delta x$ is of higher order than $z_{ij} \cdot \Delta y \cdot \Delta x$. Therefore

$$V = \lim_{\substack{m,\, n \to \infty \\ \Delta y \to 0 \\ \Delta x \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} z_{ij} \Delta y \cdot \Delta x = \int_a^b \int_{y_1}^{y_2} f(x, y) dy\, dx.$$

The rectangular column used may be taken perpendicular to any other plane of reference and the order of integration may be reversed if the operations are thereby simplified for certain solids.

When the integration is performed with respect to y, the x_j and Δx are held constant so that the part of the required volume between the planes $x = x_j$ and $x = x_j + \Delta x$ is evaluated. The result of this integration is an expression under the remaining integral sign for the *slice* shown in Fig. 189. The second integration is the limit of the sum of such slices as $\Delta x \to 0$, and this limit is the required volume.

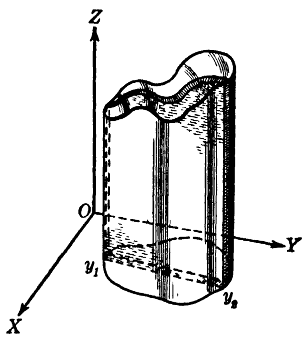


FIG. 189

EXAMPLES

1. Find the volume under the surface $2z = x^2 + y^2$ which is above the xy plane and inside the cylinder $x^2 + y^2 = 4$.

SOLUTION. Any element $\Delta y \cdot \Delta x$ in the xy plane supports a column represented by

$$\Delta V_{ij} = z_{ij} \cdot \Delta y \cdot \Delta x + k \cdot \Delta z_{ij} \cdot \Delta y \cdot \Delta x, \qquad |k| \leqq 1.$$

Therefore

$$V = \lim_{\substack{m,\, n \to \infty \\ \Delta y \to 0 \\ \Delta x \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} z_{ij} \Delta y \cdot \Delta x.$$

Considering y_i variable, the limit of the sum of columns for x_j and Δx constant is taken from the xz plane to the cylindrical surface $y = \sqrt{4 - x^2}$ for that part of the volume in the first octant. This first integration gives a slice as shown in Fig. 190. To find the total volume in the first octant we treat such slices as elements and find the limit of their sum from the yz plane to the extreme point on the x axis and the circular cylinder as $\Delta x \to 0$. That is, if we substitute for z_{ij} its value in terms of x and y, we have

$$V = \lim_{\substack{m,\, n \to \infty \\ \Delta y \to 0 \\ \Delta x \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{1}{2} (x_j^2 + y_i^2) \Delta y \cdot \Delta x.$$

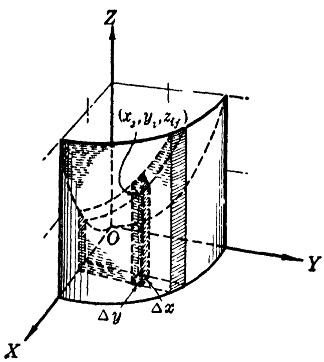


FIG. 190

Therefore

$$V = 2 \int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) dy\, dx$$

$$= 2 \int_0^2 \left(x^2 y + \frac{y^3}{3} \right) \Big]_0^{\sqrt{4-x^2}} dx$$

$$= 2 \int_0^2 \left[x^2 \sqrt{4 - x^2} + (4 - x^2) \frac{\sqrt{4 - x^2}}{3} \right] dx$$

$$= \frac{4}{3} \int_0^2 (x^2 \sqrt{4 - x^2} + 2\sqrt{4 - x^2})\, dx$$

$$= \frac{4}{3} \left[6 \sin^{-1} \frac{x}{2} + \frac{1}{4} (x^3 + 2\,x) \sqrt{4 - x^2} \right]_0^2$$

$$= 4\,\pi \text{ cubic units.}$$

2. Find the volume bounded by the plane $z = x$ and the paraboloid of revolution $z = x^2 + y^2$.

SOLUTION. Each column of cross-section $\Delta y \cdot \Delta x$ is considered as beginning at the paraboloid and extending up to the plane. The length of such a column is

$$(z_2 + k_2 \Delta z_2)_{ij} - (z_1 + k_1 \Delta z_1)_{ij}, \quad |k_2|, |k_1| \leqq 1,$$

where z_2 and z_1 represent the z coordinates of the points of the two bounding surfaces whose projections coincide at any point (x_j, y_i) below the desired volume.

Figure 191 shows the first octant and for this part of the volume the integration with respect to y is from $y = 0$ to the intersection of the two surfaces,

which is located where $x = x^2 + y^2$ or $y = \sqrt{x - x^2}$. The limits for x are determined by the points common to $z = x$, $z = x^2 + y^2$, and $y = 0$, which are (0, 0, 0) and (1, 0, 1). Therefore, the symmetry of the solid gives

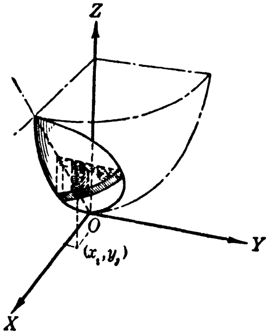


FIG. 191

$$V = \lim_{\substack{m,n\to\infty \\ \Delta y\to 0 \\ \Delta x\to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n} [x_i - (x_i^2 + y_j^2)]\Delta y\cdot\Delta x$$

$$= 2\int_0^1\int_0^{\sqrt{x-x^2}} (x - x^2 - y^2)dy\,dx$$

$$= 2\int_0^1 \left(xy - x^2y - \frac{y^3}{3}\right)\Big]_0^{\sqrt{x-x^2}} dx$$

$$= \frac{4}{3}\int_0^1 (x - x^2)^{3/2}dx.$$

Completing the square inside the parenthesis and setting $x - 1/2 = (1/2)\sin\theta$, we have

$$V = \frac{4}{3}\int_{-\pi/2}^{\pi/2} \frac{1}{16}\cos^4\theta\,d\theta$$

$$= \frac{1}{48}\left[\frac{3\,\theta}{2} + \sin 2\,\theta + \frac{\sin 4\,\theta}{8}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{\pi}{32} \text{ cubic units.}$$

170. Volumes Defined by Cylindrical Coordinates. The problem of evaluating the volume bounded by surfaces is in general greatly simplified if the surfaces are symmetric with respect to some coordinate axis so that cylindrical coordinates may be used. Then we divide the volume into elements by means of coaxial cylinders around and planes through the axis of symmetry. The cross-section of such a column is the same as the element of area in polar coordinates so that we may use $r_i\cdot\Delta r\cdot\Delta\theta$ to represent it. Then if the z axis is the axis of symmetry we have

$$\Delta V_{ij} = (z_2 + k_2\cdot\Delta z_2 - z_1 - k_1\cdot\Delta z_1)_{ij} r_i\cdot\Delta r\cdot\Delta\theta,$$
$$|k_2|,\ |k_1| \leqq 1.$$

Therefore

$$V = \lim_{\substack{m,n\to\infty \\ \Delta r\to 0 \\ \Delta\theta\to 0}} \sum_{j=1}^{m}\sum_{i=1}^{n} (z_2 - z_1)_{ij} r_i\cdot\Delta r\cdot\Delta\theta$$

$$= \int_\alpha^\beta\int_0^{f(\theta)} (z_2 - z_1)r\,dr\,d\theta,$$

where V is the volume between the two surfaces $z = f_1(r)$ and $z = f_2(r)$.

EXAMPLES

1. Find the volume required in Example 1 of the preceding article.

SOLUTION. The paraboloid is $2z = r^2$ and the cylinder is $r = 2$. Therefore, since $z_2 = r^2/2$ and $z_1 = 0$,

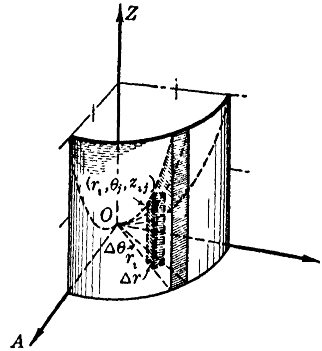


FIG. 192

$$\begin{aligned} V &= \lim_{\substack{m,\, n \to \infty \\ \Delta r \to 0 \\ \Delta\theta \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} z_{ij} r_i \cdot \Delta r \cdot \Delta\theta \\ &= 4\int_0^{\pi/2}\int_0^2 \frac{r^2}{2} r\,dr\,d\theta \\ &= 2\int_0^{\pi/2} \left(\frac{r^4}{4}\right)\Big]_0^2 d\theta = 8\int_0^{\pi/2} d\theta \\ &= 4\pi \text{ cubic units.} \end{aligned}$$

2. The second example of the preceding article represents a solid bounded by the paraboloid $z = r^2$ and by the plane $z = r\cos\theta$. Setting the two values of z equal, we find from $r^2 = r\cos\theta$, that 0 and $\cos\theta$ are the limits for r. Therefore

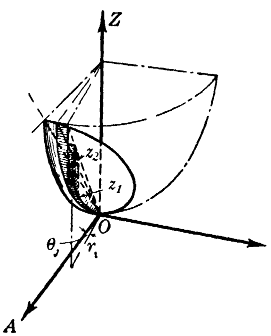


FIG. 193

$$\begin{aligned} V &= \lim_{\substack{m,\, n \to \infty \\ \Delta r \to 0 \\ \Delta\theta \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} (r_i \cos\theta_j - r_i^2) r_i \cdot \Delta r \cdot \Delta\theta \\ &= 2\int_0^{\pi/2}\int_0^{\cos\theta} (r^2\cos\theta - r^3)\,dr\,d\theta \\ &= 2\int_0^{\pi/2} \left[\frac{r^3\cos\theta}{3} - \frac{r^4}{4}\right]_0^{\cos\theta} d\theta \\ &= \frac{1}{6}\int_0^{\pi/2} \cos^4\theta\,d\theta \\ &= \frac{1}{24}\left[\frac{3\theta}{2} + \sin 2\theta + \frac{\sin 4\theta}{8}\right]_0^{\pi/2} \\ &= \frac{\pi}{32} \text{ cubic units.} \end{aligned}$$

The first of these examples is very greatly simplified by the change of coordinates and the second is simplified in the integration. In many problems the student must decide whether to use rectangular coordinates, or cylindrical coordinates. Hence it is important to know the advantages of each system.

171. Special Volumes by Double Integration. It is advisable to use double integration when considering a solid whose cross-sectional area is not readily expressed in terms of the variable of integration. This method is illustrated in the solution of the following example.

EXAMPLE

What is the volume of a solid with circular base whose sections perpendicular to a given diameter of the base are parabolic segments with the height of each equal to its base?

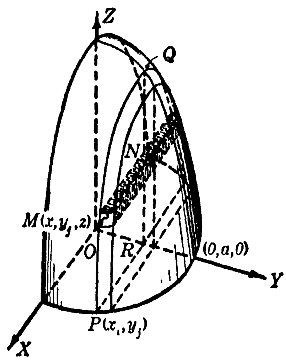


FIG. 194

SOLUTION. Take the base in the xy plane with the given diameter as the y axis. The variable of integration for single integration would then be y and the student will probably have some trouble expressing the slice perpendicular to the y axis in terms of y. The section PQR is in the plane $y = y_j$. Then $2\,RP = RQ = 2\,x_\iota$, by hypothesis. With x and z variable in the plane $y = y_j$ we have $NM = x$ and $NQ = 2\,x_\iota - z$. Therefore the parabolic segment makes

$$\frac{(2\,x)^2}{(2\,x_\iota)^2} = \frac{2\,x_\iota - z}{2\,x_\iota},$$

or

$$2\,x = \sqrt{4\,x_\iota^2 - 2\,x_\iota z}.$$

The volume of the horizontal element shown in Fig. 194 is therefore

$$\Delta V_{lj} = 2\left(\frac{\sqrt{4\,x_\iota^2 - 2\,x_\iota z}}{2} + k\cdot\Delta x\right)\Delta z\cdot\Delta y,\ 0 < k < 1,$$

whence, since $x_\iota^2 + y_j^2 = a^2$,

$$V = \lim_{\substack{m,n\to\infty\\ \Delta z\to 0\\ \Delta y\to 0}} \sum_{j=1}^{m}\sum_{l=1}^{n}\Delta V_{lj}$$

$$= 2\int_0^a \int_0^{2\sqrt{a^2-y^2}} \sqrt{4\,a^2 - 4\,y^2 - 2\,z\sqrt{a^2 - y^2}}\,dz\,dy$$

$$= 2\int_0^a \left[-\frac{(4\,a^2 - 4\,y^2 - 2\,z\sqrt{a^2-y^2})^{3/2}}{3\sqrt{a^2-y^2}}\right]_0^{2\sqrt{a^2-y^2}} dy$$

$$= 2\int_0^a 8\left(\frac{a^2 - y^2}{3}\right)dy = \frac{16}{3}\left(a^2y - \frac{y^3}{3}\right)\Big]_0^a$$

$$= \frac{32\,a^3}{9} \text{ cubic units.}$$

PROBLEMS

Use an element such that the volumes bounded as below may be found by double integration. Write the equations where not given.

1. A sphere. *Ans.* $4\,\pi a^3/3$ cu. units.

2. An ellipsoid of revolution.

3. That part of the cylinder $r = 2 \sin \theta$ between the planes $z = 0$ and $2z = 4 + r \sin \theta$. *Ans.* $5\pi/2$ cu. units.

4. Below by $z = 0$, above by the cone $r = z$, laterally by the cylinder $r = 2 \sin \theta$.

5. The plane $z = 8$ and the cone $z = 2r$. *Ans.* $128\pi/3$ cu. units.

6. That part of the cylinder $r = \cos \theta$ between the cone $z = r$ and the plane $z = 0$.

7. The paraboloid $4x^2 + y^2 = 4z$ and the plane $z = 4$. *Ans.* 16π cu. units.

8. The paraboloid $y^2 + z^2 = 2x$ and plane $x + y = 1$.

9. The plane $z = 0$, and the cylinders $x^2 + y^2 = 4$ and $x^2 = 4 - 2z$. *Ans.* 6π cu. units.

10. The paraboloid $4x^2 + z^2 = 4y$ and plane $y = 2$.

11. In the first quadrant by the paraboloid $5 - z = x^2 + 4y^2$ and the planes $x = y$, $z = 0$, and $x = 0$. *Ans.* $(25/8) \sin^{-1}(1/\sqrt{5})$ cu. units.

12. The cylinder $r = a(1 + \cos \theta)$ and the planes $z = 0$ and $z = r \cos \theta$.

13. The cylinder $r = a \sin \theta$ and the planes $r \cos \theta + z = a$ and $\theta = \pi/2$, $z = 0$. *Ans.* $a^3(3\pi - 2)/24$ cu. units.

14. The plane $z = 0$ and the cylinders $x^2 + y^2 = 1$ and $x^2 = 4 - z$.

15. The cylinders $r = \sqrt{\cos \theta}$, $r^2 \cos^2 \theta = 1 - z$, and the plane $z = 0$. *Ans.* $1 - 3\pi/32$ cu. units.

16. The paraboloid $x^2 + y^2 = 4z$, the cylinder $x^2 + y^2 = 4x$, and the plane $z = 0$.

17. The cylinder $y = x^2$ and the planes $x + y + z = 2$, $x = 0$, and $z = 0$ in the first octant. *Ans.* $17/20$ cu. units.

18. The planes $z = \pm 1$ and the cone $z^2 = 8x^2 + y^2$.

19. The paraboloid $y + 4x^2 + 9z^2 = 0$ and the plane $y + 1 = 0$. *Ans.* $\pi/12$ cu. units.

20. The dome $z = a - r^3/a^2$ and the plane $z = 0$.

21. $(x/a)^{1/2} + (y/b)^{1/2} + (z/c)^{1/2} = 1$ and the coordinate planes. *Ans.* $abc/90$ cu. units.

22. A circular paraboloid of altitude a units and radius of base b units.

172. Volumes by Triple Integration. The evaluation of the volume of a solid bounded by surfaces whose equations are known may be done by ***triple integration.*** If given in rectangular coordi-

nates, assume the desired volume divided by three sets of parallel planes, one set parallel to the yz plane and Δx apart, another parallel to the xz plane and Δy apart, and the third set parallel to the xy plane, Δz apart. These planes divide the solid into small cubical blocks $\Delta z \cdot \Delta y \cdot \Delta x$ in volume together with the irregular blocks that do not lie wholly within the solid. The sum of the volumes of those blocks which lie wholly within the solid is an approximation for the desired volume. This approximation is better the nearer the planes of each set are to each other. Also the limit of such a sum is the volume of the solid if the dimensions of the blocks approach zero as a limit. That is,

$$V = \lim_{\substack{\Delta z \to 0 \\ \Delta y \to 0 \\ \Delta x \to 0}} \sum\sum\sum \Delta z \cdot \Delta y \cdot \Delta x = \int_a^b \int_{y_1}^{y_2} \int_{z_1}^{z_2} dz\, dy\, dx,$$

where z_1 and z_2 may be functions of x and y or of either or constants; y_1 and y_2 may be functions of x or constants; the limits of x are *necessarily constants.*

If cylindrical coordinates are used, similar reasoning gives the volume as expressed by

$$V = \lim_{\substack{l,\, m,\, n \to \infty \\ \Delta z \to 0 \\ \Delta r \to 0 \\ \Delta\theta \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{l} r_i \cdot \Delta z \cdot \Delta r \cdot \Delta\theta$$

$$= \int_\alpha^\beta \int_{r_1}^{r_2} \int_{z_1}^{z_2} r\, dz\, dr\, d\theta,$$

where the limits of z may be functions of r and θ; those for r may be functions of θ; those for θ are *necessarily constants.*

EXAMPLES

1. Find the volume requested in Example 1 of § 169.

SOLUTIONS. (*a*) RECTANGULAR COORDINATES. Here $\Delta V = \Delta z \cdot \Delta y \cdot \Delta x$. Also the limits of z are the paraboloid and the xy plane, or $z_1 = 0$, and $z_2 = (x_i^2 + y_j^2)/2$. The limits of y for one-fourth of the volume are from the xz plane to the cylinder, or $y_1 = 0$, and $y_2 = \sqrt{4 - x_i^2}$. The extreme values of x for the first octant are 0 and 2. Therefore

$$V = \lim_{\substack{\Delta z \to 0 \\ \Delta y \to 0 \\ \Delta x \to 0}} \Sigma\Sigma\Sigma\, \Delta z \cdot \Delta y \cdot \Delta x = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{(x^2+y^2)/2} dz\, dy\, dx.$$

Integrating this with respect to z, we get the expression for the volume as a double integral, as given in the article mentioned above.

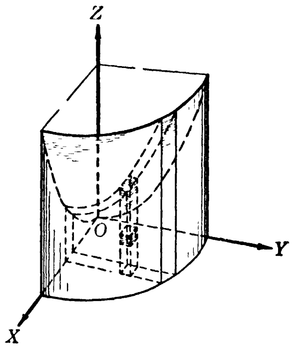


FIG. 195

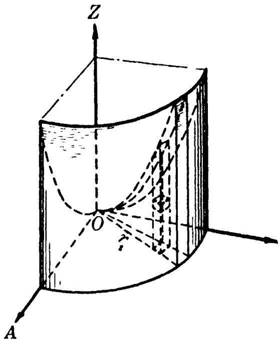


FIG. 196

(b) CYLINDRICAL COORDINATES. See Fig. 196. In this representation $z_1 = 0$, $z_2 = r_i^2/2$ and $r_1 = 0$, $r_2 = 2$, while $\theta_1 = 0$ and $\theta_2 = \pi/2$ gives one-fourth of the volume. Therefore

$$V = \lim_{\substack{l,\, m,\, n \to \infty \\ \Delta z \to 0 \\ \Delta r \to 0 \\ \Delta\theta \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{l} r_i \cdot \Delta z \cdot \Delta r \cdot \Delta\theta$$

$$= 4 \int_0^{\pi/2} \int_0^2 \int_0^{r^2/2} r \, dz \, dr \, d\theta,$$

and integration with respect to z gives the double integral used in § 170.

These examples should make the student realize that if we use fewer integrations than the number of dimensions of the quantity in question, we assume one or more integrations in so doing. That is, we choose as an element of the quantity one that is composed of more elemental units and thereby assume integrations. Therefore, *areas* and any quantity that depends upon *two dimensions* may be evaluated by either *single* or *double integration; volumes* and *three dimensional quantities* may be evaluated by *single, double,* or *triple integration. After one integration of a triple integral the resulting expression is the same one which would have been formed to solve the problem by a double integral. After another integration the result is just the form necessary to solve the problem by a single integral.* The proper choice of limits will always produce this result.

2. Find the volume bounded by the paraboloid $z = x^2 + 4y^2$ and the ylinder $z = 6 - 3y^2$.

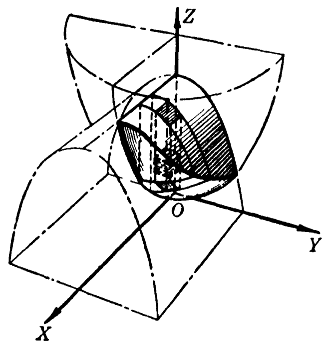


FIG. 197

SOLUTION. The element of volume is

$$\Delta V = \Delta z \cdot \Delta y \cdot \Delta x.$$

Hence we have

$$V = \lim_{\substack{\Delta z \to 0 \\ \Delta y \to 0 \\ \Delta x \to 0}} \Sigma\Sigma\Sigma\, \Delta z \cdot \Delta y \cdot \Delta x$$

$$= 4 \int_0^{\sqrt{6}} \int_0^{\sqrt{(6-x^2)/7}} \int_{x^2+4y^2}^{6-3y^2} dz\, dy\, dx.$$

The limits of y are found by setting the values of z for the two surfaces equal. Thus, $x^2 + 4y^2 = 6 - 3y^2$, so $7y^2 = 6 - x^2$, or $y = \pm\sqrt{(6 - x^2)/7}$. Then the limits for x are found from this relation between x and y when $y = 0$. The first integration gives

$$V = 4 \int_0^{\sqrt{6}} \int_0^{\sqrt{(6-x^2)/7}} (6 - x^2 - 7y^2) dy\, dx.$$

Therefore

$$V = \frac{4}{\sqrt{7}} \int_0^{\sqrt{6}} \left[6\sqrt{6 - x^2} - x^2\sqrt{6 - x^2} - \frac{(6 - x^2)^{3/2}}{3} \right] dx$$

$$= \frac{8}{3\sqrt{7}} \int_0^{\sqrt{6}} (6 - x^2)^{3/2}\, dx.$$

Let $x = \sqrt{6} \sin\theta$, so that $dx = \sqrt{6}\cos\theta\, d\theta$ and $(6 - x^2)^{1/2} = \sqrt{6}\cos\theta$. Then

$$V = \frac{96}{\sqrt{7}} \int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{24}{\sqrt{7}} \left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta \right]_0^{\pi/2}$$

$$= \frac{18\pi}{\sqrt{7}} \text{ cubic units.}$$

173. Mass. If ρ represents the density of a solid at a point, the element of mass in rectangular coordinates is $\rho \cdot \Delta z \cdot \Delta y \cdot \Delta x$. When the solid is homogeneous ρ is a constant and should be placed in front of the integral; however, for a variable density ρ must be replaced by some function of the coordinates before any integration is performed.

In cylindrical coordinates the element of mass ΔM is $\rho \cdot r_i \cdot \Delta z \cdot \Delta r \cdot \Delta\theta$.

PROBLEMS

Use triple integration to find the volumes bounded by the following surfaces. Show a figure for each. Write any necessary equations. (Nos. 1–7.)

1. The plane $z = 4$ and the paraboloid $z = r^2$. *Ans.* 8π cu. units.

2. The cylinder $r = a \cos \theta$ and the sphere $r^2 + z^2 = a^2$.

3. The paraboloid $2z = 4 - r^2$ and above the cone $6z = 5r$.
Ans. $352\pi/3^5$ cu. units.

4. That part of the cylinder $r = 2 \cos \theta$ between the paraboloid $r^2 = 9 - 3z$ above and the plane $z = 0$ below.

5. That part of the sphere $r^2 + z^2 = a^2$ inside the cylinder $r^2 = a^2 \cos 2\theta$.
Ans. $(2/9)(20 + 3\pi - 16\sqrt{2})a^3$ cu. units.

6. That part of the cylinder $r = 4 \sin \theta$ bounded below by $z = 0$ and above by the cone $r = 2z$.

7. Inside the sphere $r^2 + z^2 = 4$ and outside the cylinder $r = 2 \sin \theta$.
Ans. $16(3\pi + 4)/9$ cu. units.

Name each of the following surfaces, sketch it, and find the desired volume. (Nos. 8–22.)

8. Below $y^2 = 4 - 2z$, above $z = 0$, and within $x^2 + y^2 = 4$.

9. Bounded above by $r^2 + z^2 = 4$, below by $z = r$.
Ans. $8\pi(2 - \sqrt{2})/3$ cu. units.

10. Bounded above by $r^2 + z^2 = 5$, below by $r^2 = 4z$.

11. Enclosed by $x^2 + y^2 = 2z$ and $y = z - 1$. *Ans.* $9\pi/4$ cu. units.

12. Bounded above by $r^2 + z^2 = 2z$, below by $r^2 = z$.

13. Inside the cylinder $r = 2 \cos 2\theta$, outside the cone $z = 2r$, and above $z = 0$. *Ans.* $128/9$ cu. units.

14. The part of $r^2 + z^2 = a^2$ inside $r = a \cos 2\theta$.

15. Bounded below by $r = 3z$, above by a sphere of radius 2 units with its center at the pole. *Ans.* $(16\pi/3)(1 - 1/\sqrt{10})$ cu. units.

16. The larger volume bounded by a cone with a vertical angle of 90° at the center of a sphere, and the sphere.

17. Bounded by $r = \cos \theta$, $z = 0$, and $z = 2r \cos \theta$. *Ans.* $\pi/4$ cu. units.

18. Bounded by $z = 0$, $r = \cos \theta$, and $z = r$, with $z > 0$.

19. Bounded by $z = 0$, $r^2 = 4z$, and $r = 4 \cos \theta$. *Ans.* 6π cu. units.

20. Bounded by $x^2 + 4z^2 = 8y$, and $y = 1$.

21. Ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. *Ans.* $4\,\pi abc/3$ cu. units.

22. Bounded by $z = 0$, $x^2 + y^2 = az$, and $x^2 + y^2 = ax$.

23. Three equal cylinders of radius a units have their axes along three mutually perpendicular lines which meet in a point. Find the volume common to the three cylinders. *Ans.* $8(2 - \sqrt{2})a^3$ cu. units.

24. Find the volume enclosed by the surface of $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

25. The density of a cone at a point varies as the square of the distance of the point from the axis. If the cone has a radius of a and altitude h, find its mass. *Ans.* $\pi ka^4h/10$ units.

Use triple integration to find the masses of the solids bounded as follows. (Nos. 26–32.)

26. A hemisphere, if ρ is proportional to the distance of any point from the base.

27. A cone of radius 1 unit and altitude 2 units if the density at every point is equal to the square of its distance from the vertex. Change the order of integration and the limits, and check. *Ans.* $9\,\pi/5$ units.

28. An ellipsoid of revolution, if ρ equals the square of the distance of any point from the axis of revolution.

29. A right circular cylinder, if the density at any point is proportional to the square of its distance from a diameter of the base.

Ans. $k\pi a^2h(4\,h^2 + 3\,a^2)/12$ units.

30. The same as Problem 29 for a right circular cone.

31. Bounded by $z = r^2$ and $z = 4$, if ρ is proportional to the square of the distance of any point from the line $z = 0$, $\theta = \pi/2$. *Ans.* $208\,\pi k/3$ units.

32. Bounded by $z = x^2 + 4\,y^2$ and $z = 4$, if ρ is proportional to the distance of any point from the y axis.

174. Moment of Inertia. The ***moment of inertia*** of a particle of mass m ***about a point*** is defined as the product of the mass m by the square of the distance R from the point to the particle. Thus, $I = mR^2$. Also the moment of inertia of an aggregate of particles about a point is the sum of the moments of inertia of all of the particles about the point.

Consider now a thin plate of metal of uniform density and thickness. If we have its boundary given by $y = f(x)$ and divide the plate into small elements Δy by Δx and take $P(x_i, y_i)$ a point of each element, the moment of inertia of any element about the origin is approximately $\rho(x_i^2 + y_i^2)\cdot\Delta y\cdot\Delta x$, where ρ is the mass

per unit of area of the plate. Hence the moment of inertia of the whole plate about the origin is

$$(1) \qquad I_o = \lim_{\substack{m, n \to \infty \\ \Delta y \to 0 \\ \Delta x \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_i^2 + y_j^2)\Delta y \cdot \Delta x$$

$$= \rho \int_a^b \int_{y_1}^{y_2} (x^2 + y^2) dy\, dx,$$

where the limits of integration are taken so as to cover the area of the plate.

The ***moment of inertia about an axis*** is similarly defined with R representing the distance of a particle from the given axis. Thus the relation (1) for the x and y axes gives

$$(2) \qquad I_x = \rho \int_a^b \int_{y_1}^{y_2} y^2\, dy\, dx,$$

$$I_y = \rho \int_a^b \int_{y_1}^{y_2} x^2\, dy\, dx.$$

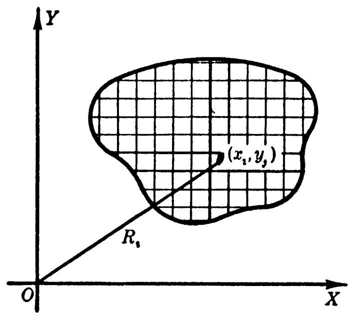


FIG. 198

The moment of inertia of a thin plate about a point of its plane is the sum of its moments of inertia about two perpendicular lines of the plane through the point. This follows immediately from (1) and (2) since $I_o = I_x + I_y$.

If the density is not uniform we have $\rho = F(x, y)$ which must remain a part of the integrand.

These definitions permit us to find the moment of inertia about a point (***polar moment***), or about an axis, of a thin plate (*area*), a small wire (*arc*), or a solid (*volume*). In each case, we divide the mass M of the whole into elements ΔM_i and define I as follows:

$$(3) \qquad I = \lim_{\substack{n \to \infty \\ \Delta M_i \to 0}} \sum_{i=1}^{n} R_i^2 \cdot \Delta M_i = \int R^2\, dM,$$

where limits are taken so as to cover the whole quantity under consideration. Of course for areas and volumes double and triple

integration respectively are implied in the definition. The student must be careful to represent by ΔM_i an element of any quantity being considered multiplied by its mass per unit of the quantity. Also be sure that R_i is the distance of this element from the point or line with respect to which the moment of inertia is desired.

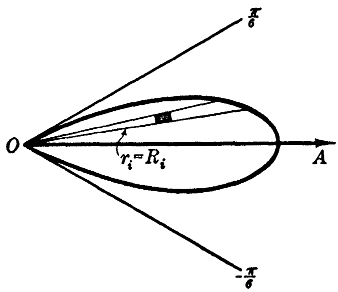


FIG. 199

EXAMPLES

1. Find the moment of inertia about the pole of a thin uniform plate in the shape of one loop of the curve whose equation is $r = a \cos 3\theta$.

SOLUTION. Here $\Delta M_i = \rho \Delta A_i$, and ΔA_i we have seen is $r_i \cdot \Delta r \cdot \Delta\theta$ except for infinitesimals of higher order. Also $R_i = r_i$ and ρ is assumed constant. Hence

$$I_0 = \lim_{\substack{m,n \to \infty \\ \Delta r \to 0 \\ \Delta\theta \to 0}} \sum_{j=1}^{m} \sum_{i=1}^{n} \rho \cdot r_i^2 (r_i \cdot \Delta r \cdot \Delta\theta)$$

$$= \rho \int_{-\pi/6}^{\pi/6} \int_0^{a \cos 3\theta} r^3\, dr\, d\theta$$

$$= \frac{\rho\, a^4}{4} \int_{-\pi/6}^{\pi/6} \cos^4 3\theta\, d\theta$$

$$= \frac{\rho\, a^4}{4} \int_{-\pi/6}^{\pi/6} \frac{1}{4}\left[1 + 2\cos 6\theta + \frac{1}{2} + \frac{1}{2}\cos 12\theta\right] d\theta$$

$$= \frac{\rho\, a^4}{16}\left[\frac{3\theta}{2} + \frac{\sin 6\theta}{3} + \frac{\sin 12\theta}{24}\right]_{-\pi/6}^{\pi/6}$$

$$= \frac{\rho\pi\, a^4}{32} \text{ units.}$$

2. Set up the moment of inertia about the polar axis of the solid bounded by $z = r^2$ and $z = a$.

SOLUTION. In this case, $R_{ij}^2 = r_i^2 \sin^2\theta_j + z_{ij}^2$. Since z varies from the paraboloid to the plane, r from the z axis to the intersection of the paraboloid and the plane where it is $\sqrt{a}$, and θ from 0 to 2π, we have

$$I = 4\int_0^{\pi/2} \int_0^{\sqrt{a}} \int_{r^2}^{a} \rho(r^2 \sin^2\theta + z^2) r\, dz\, dr\, d\theta.$$

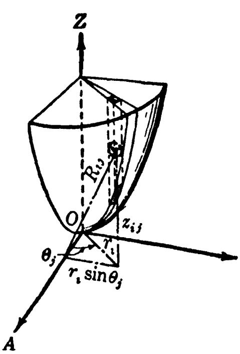


FIG. 200

The integrations are readily performed.

PROBLEMS

Find the moments of inertia of each of the areas bounded as follows. (Nos. 1–10.)

1. $xy = 12$, $y = 0$, $x = 2$, $x = 4$, about $x = 0$. *Ans.* $72\,\rho$ units.

2. $y^2 = 2(x - 4)$, $x = 3\,y$, about $x = 0$.

3. A loop of the four-leafed rose $r = a \cos 2\,\theta$, about the pole.
Ans. $3\,\pi a^4 \rho/64$ units.

4. The loop of $y^2 = x^2(2 - x)$, about $x = 0$.

5. A semicircle, about (*a*) the tangent parallel to its diameter; (*b*) a tangent perpendicular to the diameter. *Ans.* (*a*) $a^4\rho(15\,\pi - 32)/24$ units.

6. $3\,y^2 = 4\,x$, $3\,y = 2\,x$, about $y = 0$.

7. $x^2 = y$, $y = x + 6$, about $y = 0$. *Ans.* $415\frac{5}{28}\,\rho$ units.

8. The smaller area bounded by $y = \cos x$, $y = 0$ and $x = \pi/4$, about $x = 0$.

9. $y^2 = 1 - x$, $x = 0$, about $x = 0$. *Ans.* $32\,\rho/105$ units.

10. $r = 4 \sin \theta$, (*a*) about the pole; (*b*) about $\theta = 0$.

Find the moments of inertia of the areas and solids bounded as follows. (Nos. 11–20.)

11. A semicircle of radius 2 units, about a line parallel to its diameter and 3 units from the diameter (two cases). *Ans.* $4(5\,\pi \pm 8)\,\rho$ units.

12. Area in the first quadrant inside $r = 2\,a \sin 2\,\theta$ and outside $r = a$, about $\theta = \pi/2$.

13. Area in $x^2 + y^2 = 2$ above $x^2 = y$, about the y axis.
Ans. $\rho(5\,\pi - 8)/20$ units.

14. Area enclosed by $y = 2\,x - x^2$ and $y = 0$, about $y = 4$ if the density at a point is proportional to the distance of the point from $y = 4$.

15. The semicircle $y = +\sqrt{a^2 - x^2}$, about $y = 2\,a$.
Ans. $a^4\rho(51\,\pi - 64)/24$ units.

16. The area $y = 0$, $y = e^{2x}$, $x = 0$, $x = 1$, about $x = 0$.

17. The solid bounded by the cone $z = r$ and the plane $z = 4$, about the polar axis. *Ans.* $256\,\pi\rho$ units.

18. The solid $z = x^2 + y^2$, $z = x$, about the x axis.

19. A right circular cone of height h units and radius of base a units, about its axis. *Ans.* $\pi a^4 h\rho/10$ units.

20. The right circular cone of Problem 19, about its vertex.

21. The slant height being given, find the right circular cone having the maximum moment of inertia about its axis. *Ans.* $r = 2\,h$.

22. Find the moment of inertia of the solid bounded by $z = r^2$ and $z = 4$, about the line $\theta = \pi/2$ in the polar plane.

23. Find I_z for the solid bounded by $r = 4\cos\theta$, $z = 0$, and $z = 3$.
Ans. $36\,\pi\rho$ units.

24. The density at any point of the hemisphere $z = +\sqrt{a^2 - r^2}$ is k times the distance of the point from the circular base. Find I_z for the hemisphere. Find I_z for the part of the hemisphere within the cylinder $r = a\cos\theta$.

ADDITIONAL PROBLEMS

Find the areas bounded as follows. (Nos. 1–5.)

1. One loop of the four-leafed rose $r = 2\sin 2\,\theta$. *Ans.* $\pi/2$ sq. units.

2. Outside the circle $r = a$ and inside the cardioid $r = a(1 + \cos\theta)$.

3. Inside $r = \sin\theta$ and outside $r = \sin^2\theta$. *Ans.* $\pi/16$ sq. units.

4. Bounded by $y = e^{2x}$, $y = 0$, $x = 1$, and $x = 2$.

5. Bounded by the ellipse $4\,x^2 + 6\,y^2 = 9$. *Ans.* $9\,\pi/2\sqrt{6}$ sq. units.

Find the volumes of revolution generated as follows. (Nos. 6–11.)

6. $y = x^2$, $x = 1$, $y = 0$, (a) about $x = 0$; (b) about $y = 0$.

7. $y^2 = kx$, $x = 0$, $y = a$, about $x = 0$. *Ans.* $\pi a^5/5\,k^2$ cu. units.

8. $x^2 + 4 = 4\,y$, $x = 0$, $y = x$, about the x axis.

9. $y = x^3$, $x = 1$, $y = 0$, about $x + 2 = 0$. *Ans.* $7\,\pi/5$ cu. units.

10. One arch of $x = \cos 4\,y$ and $x = 0$, about $x = 2$.

11. One arch of $x = \sin 2\,y$ and $x = 0$, about the y axis.
Ans. $\pi^2/4$ cu. units.

Find the following volumes. (Nos. 12–16.)

12. The volume bounded above by $r^2 + z^2 = 4$ and below by $r^2 = 2\,z$.

13. The volume of $y^2 + x^2 = 2\,z$ cut off by $2\,z + y = 6$.
Ans. $(625/64)\,\pi$ cu. units.

14. The volume bounded by $r^2 = 4\,z$, $r = 4\cos\theta$, and $z = 0$.

15. The volume of $x^2 + y^2 = az$, $x^2 + y^2 = 2\,ax$, and $z = 0$.
Ans. $3\,\pi a^3/2$ cu. units.

16. The volume under $y^2 = 4 - 2\,z$, above $z = 0$, and within the cylinder $x^2 + y^2 = 4$.

17. The volume between two surfaces is given by

$$4\int_0^{\pi/2}\int_0^{a/\sqrt{2}}\int_r^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta.$$

Write the equations of the surfaces and sketch them.

Ans. The solid lies above $z = r$, below $r^2 + z^2 = a^2$.

18. What is left of a sphere of radius 5 in. if a circular hole is cut through its center, if the diameter of the hole is 8 in.?

19. A rectangular sheet of metal is of variable density. The density along any line parallel to an end is proportional to the square of the distance of the line from the end. When the distance is 4 units, the density is 2. Find the mass of a sheet measuring 8 by 10 units. *Ans.* $333\frac{1}{3}$ units.

20. The same as Problem 19 except that the density is proportional to the cube of the distance and is 36 at a distance of 6 units.

21. Find the mass of a thin plate bounded by $y^2 = 4x$ and $x = 1$, the density at any point P being proportional to xy^2. *Ans.* $32\,k/21$ units.

Find the moment of inertia of the following areas and solids. (Nos. 22–31.)

22. The solid bounded by $z = r^2$ and $z = c$, about $\theta = 0$ in the polar plane.

23. The solid bounded above by $z = 6 - r$ and below by $z = r^2$, about $\theta = 0$ in the polar plane. *Ans.* $376\,\pi\rho/3$ units.

24. A right circular cone, about a diameter of its base.

25. The area within one loop of $r = 2\,a\cos 2\,\theta$ and outside $r = a$, about the pole. *Ans.* $a^4\rho(20\,\pi + 21\sqrt{3})/48$ units.

26. I_z of a cylinder of altitude h units with one loop of $r = a\sin 3\,\theta$ as its base, if ρ is proportional to the square of the distance of a point from $\theta = 0$ in the polar plane.

27. A paraboloid of revolution of altitude a units and radius of base b units, about its axis. *Ans.* $b^4a\pi\rho/6$ units.

28. The paraboloid of Problem 27 about a diameter of its base.

29. I_x of the mass common to $x^2 + y^2 + z^2 = 4\,a^2$ and the cylinder of radius a units with its axis along the z axis. *Ans.* $2\,a^5\pi\rho(128 - 51\sqrt{3})/15$ units.

30. The area of a loop of $r^2 = 4\sin 3\,\theta$, about the pole.

31. The area of $y = e^x$, $x = 0$, $x = 2$, $y = 0$, about the origin.

Ans. $\rho(e^6 + 18\,e^2 - 19)/9$ units.

32. What does $k\int_0^{\sqrt{3}}\int_z^{3}\int_1^{\sqrt{4-z^2}} (y^2 + z^2)\,dx\,dy\,dz$ represent?

33. Prove that the moment of inertia of an elliptical plate about either axis is one-fifth its mass times the square of the length of that semi-axis.

34. A hollow sphere has an inside radius of 6 in. and outside radius of 10 in. At any point, the density varies inversely as the distance of the point from the center; and on the outer surface, the density is $2\frac{1}{2}$. Find the mass of the sphere.

CHAPTER XV

ADDITIONAL APPLICATIONS OF THE FUNDAMENTAL THEOREM

175. Introduction. In the two preceding chapters we have shown how the integral as the limit of a sum may be used either as a single sum or as a multiple sum. The following applications involve both ideas and although the illustrative examples exemplify the methods we think advisable to use, the student will often find that several methods are open to him.

176. Area of a Surface of Revolution. A *surface of revolution* may be considered as generated by the arc of a plane curve revolved around a line in its plane. Suppose this line to be the y axis and let $z = f(y)$ from $y = a$ to $y = b$ be the arc of the curve. Divide the interval $a \leqq y \leqq b$ along the y axis into n segments of length Δy. Planes perpendicular to the axis at the points of division will divide the surface into narrow bands. Each band is generated by a Δs_i of the curve $z = f(y)$. If z_i represents the ordinate of any point of the element Δs_i, the product $2\,\pi z_i \cdot \Delta s_i$ is approximately the area of the corresponding band, and the limit of the sum of such products as $\Delta s_i \to 0$ is the area of the surface. That is,

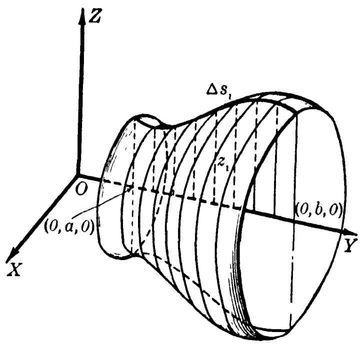


FIG. 201

$$(1) \qquad S = \lim_{\substack{n \to \infty \\ \Delta s_i \to 0}} \sum_{i=1}^{n} 2\,\pi z_i \cdot \Delta s_i$$

$$= 2\,\pi \int_{y=a}^{y=b} z\,ds$$

$$= 2\,\pi \int_a^b z\sqrt{1 + \left(\frac{dz}{dy}\right)^2}\,dy.$$

There are similar formulas for surfaces of revolution about the

other coordinate axes. Thus, if a curve in the xy plane whose equation is $y = f(x)$ is revolved about the x axis, the formula for the area of the surface is

$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{1a}$$

If the axis of rotation is any line parallel to the x axis, as $y = k$, the radius in (1a) is not y, but $y - k$. Similar formulas hold if the axis of rotation is parallel to the y axis or to the z axis.

In polar coordinates formula (1a) becomes

$$S = 2\pi \int r \sin\theta \sqrt{dr^2 + r^2 d\theta^2}, \tag{2}$$

for surfaces with the initial line as an axis. If the axis of rotation is the line $\theta = \pi/2$, the radius in (2) becomes $r \cos\theta$.

EXAMPLES

1. Find the surface of a hemisphere by considering it generated by a quadrant of a circle.

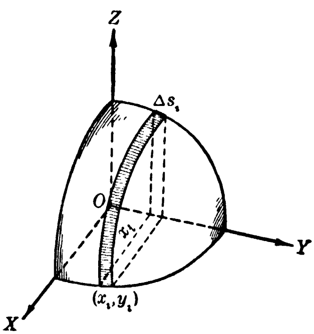


FIG. 202

SOLUTION. Suppose the circular arc has the equation $x^2 + y^2 = a^2$, $x \geqq 0$, $y \geqq 0$. Then $dx/dy = -y/x$ and the limits for y are 0 and a. Therefore

$$S = \lim_{\substack{n \to \infty \\ \Delta s_i \to 0}} \sum_{i=1}^{n} 2\pi x_i \cdot \Delta s_i = 2\pi \int_{y=0}^{y=a} x\, ds$$

$$= 2\pi \int_0^a x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_0^a x\sqrt{\frac{x^2 + y^2}{x^2}}\, dy = 2\pi a \int_0^a dy$$

$= 2\pi a^2$ square units.

2. Find the area of the surface formed by revolving one arch of the cycloid about the x axis.

SOLUTION. As $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ we have

$$dy/dx = (dy/d\theta)/(dx/d\theta) = (a\sin\theta)/[a(1 - \cos\theta)] = \text{ctn}\,(\theta/2).$$

Hence $$\sqrt{1 + (dy/dx)^2} = \sqrt{1 + \text{ctn}^2\,(\theta/2)} = \csc\,(\theta/2).$$

Therefore

$$S = \lim_{\substack{n \to \infty \\ \Delta s_i \to 0}} \sum_{i=1}^{n} 2\pi \cdot y_i \cdot \Delta s_i = 2\pi a^2 \int_0^{2\pi} (1 - \cos\theta)^2 \csc\frac{\theta}{2}\, d\theta$$

$$= 2\pi a^2 \int_0^{2\pi} 4 \sin^4\frac{\theta}{2} \csc\frac{\theta}{2}\, d\theta = 8\pi a^2 \int_0^{2\pi} \sin^3\frac{\theta}{2}\, d\theta$$

$$= 8\pi a^2 \left[-2\cos\frac{\theta}{2} + \frac{2}{3}\cos^3\frac{\theta}{2} \right]_0^{2\pi} = \frac{64\pi a^2}{3} \text{ square units.}$$

PROBLEMS

Find the area of the surface of each of the solids of revolution generated as follows. (Nos. 1–15.)

1. By revolving the parabola $y^2 = 2ax$ between the extremities of the latus rectum about $y = 0$. *Ans.* $2\pi a^2(2\sqrt{2} - 1)/3$ sq. units.

2. By revolving $x^2 = 6y$ from the origin to $y = 9/2$ about $x = 0$.

3. By revolving about the y axis, (*a*) the circle $x^2 + y^2 = 4x$, (*b*) the semicircle of this circle between $x = 0$ and $x = 2$.
Ans. (*a*) $16\pi^2$ sq. units; (*b*) $8\pi(\pi - 2)$ sq. units.

4. By revolving the *hypocycloid* $x^{2/3} + y^{2/3} = a^{2/3}$ about an axis.

5. By revolving the *cardioid* $r = a(1 + \cos\theta)$ about the polar axis.
Ans. $32\pi a^2/5$ sq. units.

6. By revolving one loop of the *lemniscate* $r^2 = a^2 \cos 2\theta$ about the polar axis.

7. By revolving one arch of $y = \sin x$ about $y = 0$.
Ans. $2\pi[\sqrt{2} + \log(1 + \sqrt{2})]$ sq. units.

8. By revolving the part of $y = \log x$ which lies in the fourth quadrant about $x = 0$.

9. By revolving the *catenary* $y = (a/2)(e^{x/a} + e^{-x/a})$ from $x = 0$ to $x = a$ about $x = 0$. *Ans.* $2\pi a^2(1 - 1/e)$ sq. units.

10. By revolving the curve $4y + x^2 - \log x^2 = 0$ from $x = 1$ to $x = 4$ about the y axis.

11. By revolving the *tractrix*, for which $dy/dx = -y/\sqrt{a^2 - y^2}$, from P_1 to P_2, about the x axis. *Ans.* $2\pi a(y_1 - y_2)$ sq. units.

12. The *torus* formed by revolving the circle $(x - a)^2 + y^2 = r^2$ about the y axis.

13. By revolving the loop of the curve $3ay^2 = x(a - x)^2$ about the x axis.
Ans. $\pi a^2/3$ sq. units.

14. By revolving the *semi-cubical parabola* $ay^2 = x^3$ from the origin to $x = 4\,a/3$ about $x = 0$.

15. By revolving the *cubical parabola* $9\,y = x^3$ from $(0, 0)$ to $(3, 3)$ about the y axis. *Ans.* $3\,\pi\,[3\sqrt{10} + \log\,(3 + \sqrt{10})]/3$ sq. units.

16. Prove that the surface produced by rotating $r^2 = a^2 \sin 2\,\theta$ about either axis, $\theta = 0$, or $\theta = \pi/2$ is equal to the surface of the circumscribed sphere.

17. Find the areas of the ellipsoids of revolution formed by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$, $(a > b)$ about the x axis; about the y axis.

Ans. $2\,\pi b[b + (a^2/c)\,\sin^{-1}(c/a)]$ sq. units;
$2\,\pi a[a + (b^2/c)\,\log\,\{(a + c)/b\}]$ sq. units.

177. Force Exerted on a Submerged Surface by a Fluid.

To find the *force* exerted on a submerged surface of variable depth we use the physical principle that the pressure at a point within a fluid is the same in all directions and is the same on units of surface of a common depth below the surface of the fluid. The force on a surface of area A submerged to the depth of h units is the weight of the column of fluid which could be supported by the area, that is, whA, where w is the weight of a cubic unit of the fluid. We now illustrate a method of applying these principles.

EXAMPLES

1. Find the force on the vertical end of a parabolic gutter if it is full of water, the gutter being 6 inches across the top and 4 inches deep.

SOLUTION. Let Fig. 203 represent an end of the gutter. Suppose it to be divided into narrow strips by means of straight lines parallel to the surface AC. Let Δh be the width of any one of these strips with parallel sides l_i and $l_i + \Delta l_i$. The area of such a strip is $l_i \cdot \Delta h + k \cdot \Delta l_i \cdot \Delta h$ (see Fig. 203), where $0 < k < 1$. Now if h_i represents the distance of the strip from the bottom of the gutter, the force on the strip is $w(l_i + k \cdot \Delta l_i)(4 - h_i)\Delta h$, approximately. The approximation is due to the fact that $(4 - h_i)$ is not the depth of all parts of the strip considered. However, the force on the strip differs from the expression approximating it by infinitesimals of higher order than the first. Therefore

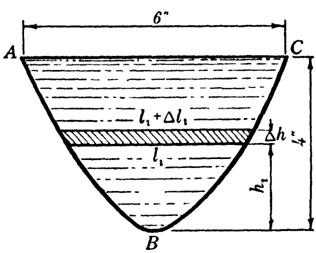


FIG. 203

$$F = \lim_{\substack{n\to\infty \\ \Delta h \to 0}} \sum_{i=1}^{n} w(l_i + k \cdot \Delta l_i)(4 - h_i)\Delta h = w \int_0^4 l(4 - h)dh.$$

To express l in terms of h, we have from the parabolic segment $l^2/36 = h/4$ or $l = 3\sqrt{h}$. Whence

$$F = 3\,w\int_0^4 (4\,h^{1/2} - h^{3/2})dh = 3\,w\left(\frac{8\,h^{3/2}}{3} - \frac{2\,h^{5/2}}{5}\right)\Bigg]_0^4$$

$$= \frac{128\,w}{5} \text{ lbs.}$$

We may also solve this example by using coordinate axes. If we choose B as origin, the coordinates of C are $(3, 4)$, and the equation of the parabola is $4\,x^2 = 9\,y$. Using any point (x, y) to set up the element of integration, we may write

$$F = w\int_0^4 2\,x\,(4 - y)dy = 3\,w\int_0^4 (4 - y)y^{1/2}dy,$$

which gives the same result as that found above.

2. A hemispherical bowl is full of water; find the total force exerted on the bowl.

SOLUTION. Consider the surface of the bowl as formed by revolving a quadrant of the circle $x^2 + z^2 = a^2$ about the z axis. Then the element Δs_i of arc generates a narrow band whose area we may represent approximately by $2\,\pi x_i \cdot \Delta s_i$. This area is at a depth of z_i and hence the approximate force on the band is $2\,\pi w x_i z_i \cdot \Delta s_i$. Therefore

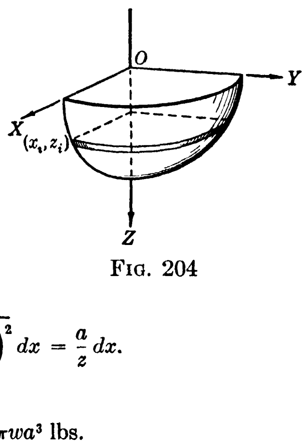


FIG. 204

$$F = \lim_{\substack{n\to\infty \\ \Delta s_i \to 0}} \sum_{i=1}^{n} 2\,\pi w x_i z_i \cdot \Delta s_i$$

$$= 2\,\pi w\int_{x=0}^{x=a} xz\,ds.$$

From $x^2 + z^2 = a^2$ we get $ds = \sqrt{1 + \left(\frac{dz}{dx}\right)^2}\,dx = \frac{a}{z}\,dx.$

Whence

$$F = 2\,\pi w a\int_0^a x\,dx = \pi w a^3 \text{ lbs.}$$

PROBLEMS

Find the force on one face of the submerged surfaces bounded or described as follows. (Nos. 1–20.)

1. A parabolic segment of altitude 12 units and base 8 units, if its axis is vertical and the vertex, above the base, is 5 units below the surface.

Ans. 3904 w/5 lbs.

2. A parabolic segment of altitude 6 units and base 12 units, vertex down and axis vertical, with the base 5 units below the surface.

3. A parabolic segment of altitude 3 units and base 4 units, plane vertical, axis horizontal and 1 unit below the surface. *Ans.* $135\,w/16$ lbs.

4. The parabolic segment of Problem 3 except that the axis is 3 units below the surface.

5. A semicircular plate, diameter horizontal, plane vertical, and tangent parallel to the diameter in the surface. *Ans.* $a^3w(3\pi - 4)/6$ lbs.

6. The same as Problem 5 with radius 2 units and diameter 8 units below the surface.

7. A circular plate, plane vertical, radius 5 units, and center 10 units below the surface. *Ans.* $250\,\pi w$ lbs.

8. The same as Problem 7 with radius 4 units and center 2 units below the surface.

9. A quarter of a circle of radius 2 units, plane vertical, one radius horizontal along the upper edge of the plate and 5 units below the surface.

Ans. $w(15\pi + 8)/3$ lbs.

10. A semicircular plate in a vertical plane, radius 4 units, diameter along the upper edge and 3 units below the surface.

11. The end of a full trough which is a right isosceles triangle with a base of 4 units. *Ans.* $8\,w/3$ lbs.

12. An isosceles triangle, plane vertical, base below the vertex. Base 6 in., sides 5 in., base 3 in. below the surface.

13. A right triangle in a vertical plane, one leg horizontal, 5 units long and 4 units below the surface. The vertical leg projects out of the liquid 2 units.

Ans. $31\frac{1}{9}\,w$ lbs.

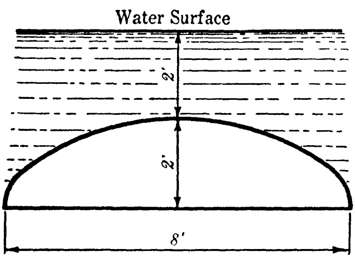


FIG. 205

14. An isosceles triangle in a vertical plane. The base is 6 units and the sides are 5 units. The base is above the vertex and 1 unit below the surface.

15. The same as Problem 14 with the base vertical and the upper end is 2 units below the surface.

Ans. $60\,w$ lbs.

16. A semi-ellipse, $a = 3$, $b = 2$ units, plane vertical, the major axis horizontal and along the lower edge. The highest point 2 units below the surface.

17. Find the force on one face of a semi-ellipse as shown in Fig. 205.

Ans. $16\,w(3\pi - 2)/3$ lbs.

18. An elliptic plate, major axis vertical, $a = 5$, $b = 2$ units, and the center is 3 units below the surface.

19. A square with sides a units has one diagonal vertical and the upper end is in the surface. Compare the force on the upper and the lower halves.

Ans. 1 to 2.

20. The same as Problem 19 but the upper end of the diagonal is b units below the surface, also find the force.

21. A horizontal cylindrical tank has a radius of 2 ft. What is the force on an end if 3 ft. of liquid is in the tank? *Ans.* $(8\pi/3 + 3\sqrt{3})\, w$ lbs.

22. A horizontal tank of 5 ft. radius is sealed and water is forced in until it is half full; the upper half of the tank is then under the pressure of one atmosphere, equivalent to 34 ft. of water. Find the force on the inside of one end due to air pressure and weight of water.

23. A trapezoidal dam has its upper base of 300 ft. in the surface, its lower base of 100 ft. is 15 ft. below the surface. Find the force. *Ans.* $585\frac{15}{16}$ tons.

24. The area $y = \sin x$, $y = 0$, $0 \leqq x \leqq \pi$ has $y = 0$ horizontal and 3 units below the surface. Find the force.

25. An oil cup is a paraboloid of revolution 6 in. in height and 2 in. radius of base at the top. If the cup is full of oil, what is the force on its curved surface? *Ans.* $0.0739\, w$ lbs.

26. An ellipsoid of revolution with semi-axes $a = 6$, $b = 4$ is half full of water. What is the force on its curved surface?

178. Work. Suppose that a body is moved a distance s along a line against a variable force, and that the force is some function of s, say $f(s)$. Now divide the distance s into small intervals Δs_i, and let $f(s_i)$ represent the force exerted at the beginning of each interval Δs_i. Then the work over each interval of the line is approximately $f(s_i) \cdot \Delta s_i$, and the sum of such products is an approximation of the work done. Hence we have

$$W = \lim_{\substack{n \to \infty \\ \Delta s_i \to 0}} \sum_{i=1}^{n} f(s_i) \cdot \Delta s_i = \int_{s=a}^{s=b} f(s)\,ds,$$

where $b - a$ represents the total distance through which the body is moved along the line.

When work is done against the force of gravity, the force exerted is the ***weight*** of the body and the distance is the ***vertical distance*** through which it is lifted.

EXAMPLES

1. If the force necessary to stretch a spring is proportional to the amount the spring is stretched and is 15 lbs. when the spring is stretched 1 inch, find the work done in stretching the spring 3 inches.

SOLUTION. We have given that $F_i = ks_i$, where s_i represents the distance the spring is stretched. Also, since $F = 15$ lbs. when $s = 1$ inch, the formula gives $k = 15$. Therefore $F_i = 15\, s_i$. Then the element of work is given by $\Delta W_i = 15\, s_i \cdot \Delta s_i$ and hence the total work is

$$W = \lim_{\substack{n \to \infty \\ \Delta s_i \to 0}} \sum_{i=1}^{n} 15\, s_i \cdot \Delta s_i = 15 \int_0^3 s\, ds = 67.5 \text{ in. lbs.}$$

2. A conical vessel is 16 ft. across the top and 12 ft. deep. If it has water in it to a depth of 10 ft., find the work necessary to pump the water to a height of 4 ft. above the top of the vessel.

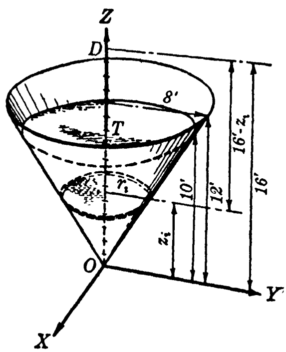


FIG. 206

SOLUTION. The work done to raise a horizontal layer of water to the point of delivery is $\Delta W_i = w(16 - z_i)\,\pi r_i^2 \cdot \Delta z_i$ approximately, where w is the weight of a cubic foot of water.

But, from similar triangles, $r_i = 2\, z_i/3$, and hence

$$W = \lim_{\substack{n \to \infty \\ \Delta z_i \to 0}} \sum_{i=1}^{n} 4\, w\pi z_i^2\, (16 - z_i) \frac{\Delta z_i}{9}$$

$$= \frac{4\, w\pi}{9} \int_0^{10} (16 - z) z^2\, dz$$

$$= \frac{4\, w\pi}{9} \left(\frac{16\, z^3}{3} - \frac{z^4}{4} \right) \Bigg]_0^{10} = \frac{34000\, \pi w}{27} \text{ ft. lbs.}$$

PROBLEMS

1. A boat moves in a straight line according to the law $s = 3\, t^2 + t$. If air and water resistance is equal to the square of its velocity, find the work done against resistance from $t = 0$ to $t = 5$ units. *Ans.* 38,480 units of work.

2. In hoisting ore from a mine, the load consists of (*a*) the weight M of the car and contents, (*b*) the weight of the cable at m lbs./ft. What work is done in hoisting a distance of h ft. from the bottom of a mine k ft. deep?

3. Find the work done in moving a body from $x = 0$ to $x = 10$ units if the force necessary at any point is $x/(100 + x^2)^{5/2}$ units.

Ans. $(4 - \sqrt{2})/12{,}000$ unit of work.

4. A body is moved in a straight line according to the laws $= 2\, t^2 + t$ against a resistance proportional to its velocity, find the work done against this resistance from $t = 0$ to $t = 3$ units.

5. The position of a body is given by $s = t \cos t$. If its motion is resisted by a force equal to its velocity, what is the work done in overcoming this resistance from $t = 0$ to $t = \pi/2$ units. *Ans.* $\pi(6 + \pi^2)/48$ units of work.

6. The same as Problem 5 if $s = t \cos t - \sin t$.

7. A tank in the form of a paraboloid of revolution, altitude 3 ft. and radius of top 2 ft., is full of water. What work is necessary to pump the water to the top? *Ans.* $6\pi w = 1178$ ft. lbs.

8. The same as Problem 7, but the water is to be pumped to a delivery point 6 ft. above the top.

9. A cistern in the form of a paraboloid of revolution, altitude 4 ft., radius of top 3 ft., contains 3 feet of water. What work is done in lowering the water level 1 ft., by pumping it to a point 2 ft. above the top of the cistern?
Ans. $19.5\,\pi w = 4875\,\pi/4$ ft. lbs.

10. A conical vessel is 4 ft. deep and the diameter of its top is 4 ft. Find the work necessary to empty the vessel, if it was full of liquid, by pumping it 2 ft. above the top of the vessel.

11. A conical vessel is full of a liquid. It is 4 ft. high and the radius of the bottom is 6 ft. What work is necessary to pump the contents to a point 2 ft. above the vertex? *Ans.* $240\,\pi w$ ft. lbs.

12. A full hemispherical bowl is 6 ft. in diameter. Find the work necessary to pump the liquid to a point 5 ft. above the top.

13. A full cylindrical tank 10 ft. long and 4 ft. across an end lies horizontally. What work is necessary to pump the contents to a point 20 ft. above the top of the tank? *Ans.* $880\,\pi w$ ft. lbs.

14. A hemispherical bowl of radius 4 ft. has its contents discharged at a point 1 ft. above the top. Find the work necessary to lower the depth of the water from 3 ft. to 2 ft.

15. A cistern filled with liquid is in the shape of a paraboloid of revolution 10 ft. deep and 6 ft. across the top. Find the work required to pump the contents to a point 4 ft. above the top. *Ans.* $330\,\pi w$ ft. lbs.

16. A trough has semi-elliptical ends. It is 2 ft. deep, 2 ft. across the top, and 8 ft. long. What work is required to pump the contents to the top if it was full at the start?

17. A cylindrical tank 10 ft. high and 10 ft. in diameter stands on a platform 50 ft. high. Find the depth of the water when one-half of the necessary work has been done to fill the tank from the ground level through a pipe in the bottom. *Ans.* 5.23 ft.

18. The weight of a body varies inversely as the square of its distance from the center of the earth, if the body is above the surface of the earth. Find the work done in lifting P lbs. of material to a height h miles above the surface.

19. A cubical tank 4 ft. across, 6 ft. deep, and 12 ft. long, is full of oil weighing 50 lbs. per cubic foot. Find how much the surface of the oil is lowered when one-third of the necessary work has been done to pump the oil to a delivery point at the top of the tank. *Ans.* $2\sqrt{3}$ ft.

20. An oil tank in the form of an inverted paraboloid of revolution is 12 ft. deep and 4 ft. across the bottom. If it is full of oil weighing 55 lbs. per cubic foot, how much is the surface lowered when one-half of the work necessary to pump the oil to the top has been done?

179. Work as Change of Kinetic Energy. If a particle of mass m moves in a straight line so that its acceleration is a, the force it exerts is defined as ma. Now since $d^2s/dt^2 = a$, and

$$\frac{d^2s}{dt^2} = \frac{d(ds/dt)}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds},$$

we have the relation

$$(1) \qquad F = mv \cdot \frac{dv}{ds}.$$

In differential form this is

$$Fds = mv\,dv,$$

and integrating from $s = s_1$ to $s = s_2$, calling v_1 and v_2 the corresponding limits for v, we find

$$\int_{s_1}^{s_2} Fds = \tfrac{1}{2}\,mv_2^2 - \tfrac{1}{2}\,mv_1^2.$$

Therefore, since $(1/2)\,mv^2$ is the ***kinetic energy*** of the particle of mass m and velocity v, the *work done in the interval* $s_1 \leqq s \leqq s_2$ *is equal to the change in the kinetic energy in the same interval.*

PROBLEMS

The following solids of given density are rotating about their axes with an angular velocity ω. Find the work each can do in coming to rest. (Nos. 1–5.)

1. A cylinder of radius a and altitude h units.
Ans. $Mv^2/2$ units. (M = mass, $v = a\omega$.)

2. A right circular cone of radius a and altitude h units.

3. A sphere of radius a units. *Ans.* $Mv^2/5$ units.

4. A hollow cylinder with inner radius a_1 and outer radius a_2, if the density at every point varies inversely as the distance from the axis.

5. A paraboloid of revolution of height h and radius of base a units.
Ans. $Mv^2/6$ units.

6. A hollow cylinder closed at both ends and a solid cylinder have the same radius, altitude, mass, and color. How can these be distinguished without taking a section?

7. A cylinder of given altitude, radius, and density is rotating with a given angular velocity. If a frictional force of 4 lbs. is applied on its surface, how many revolutions will it make before coming to rest? *Ans.* $\rho r^3 h\omega^2/32$.

8. A sphere of radius 12 inches and density $7\frac{1}{2}$ makes 15 revolutions per second about a diameter. What frictional force applied at the equator will bring the sphere to rest after 10 revolutions?

9. Assume that near the muzzle of a gun the resultant force on the base of a shell is $R(1 + kx^{-5/4})$ tons, where x is the distance in inches that the shell has traveled down the bore, and R and k are constants. What expression represents the change in the kinetic energy at the muzzle if the bore is shortened from 121 inches to 100 inches? *Ans.* $R(21 + 0.059\,k)$ in. tons.

10. The velocity of a 200 lb. sled is $60 - 4\,t$, measured in feet per second. Find the mean value of its kinetic energy as to time from $t = 0$ to $t = 10$ seconds.

180. Centroids of Plane Areas. The point at which a thin sheet of metal can be balanced is called the ***center of mass*** of the sheet; that is, it is the ***centroid*** of the area represented by the metal sheet. Since an area may be balanced at its centroid, the sum of the moments of the elements of the area about any line through its centroid must be zero. By the ***moment*** (***static moment***) of an element of area about a line we mean *the product of the number of square units in the element by its distance from the line.* Also the moment of a finite area about a line is the same as the sum of the moments of all the elements of the area about that line.

The centroid is that point at which the total area may be assumed concentrated without affecting the moment of the area about a line. Therefore, if $(\bar{x}, \bar{y})$ is the centroid, A the area, M_x the moment about the x axis, and M_y that about the y axis, we have

$$(1) \qquad M_x = A \cdot \bar{y}, \qquad M_y = A \cdot \bar{x}.$$

These equations permit us to find $\bar{x}$ and $\bar{y}$ if the area and its moments can be found.

To find M_x for a given area, we shall consider the shaded element of Fig. 207, which is parallel to the y axis. Its moment with respect to the x axis is defined as the product of its area ΔA_i, which is $(y_2 - y_1)\Delta x$, approximately, and the ordinate of its centroid, which is $(y_2 + k_2 \cdot \Delta y_2 + y_1 + k_1 \cdot \Delta y_1)_i/2$, where $|k_2|$ and $|k_1| \leqq 1$. Hence

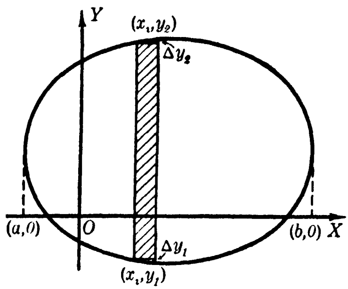


FIG. 207

$$(\Delta M_x)_i = \frac{(y_2^2 - y_1^2)_i \cdot \Delta x}{2} + e_i \cdot \Delta x,$$

where $e_i \to 0$ as $\Delta x \to 0$ if $\Delta y_1, \Delta y_2 \to 0$ also. Therefore

$$(2) \qquad M_x = \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \sum_{i=1}^{n} \frac{1}{2} (y_2^2 - y_1^2)_i \cdot \Delta x,$$

$$= \frac{1}{2}\int_a^b (y_2^2 - y_1^2)dx.$$

As the distance from the y axis to the centroid of the shaded element is $x_i + k \cdot \Delta x$, $k < 1$, we have

$$(3) \qquad M_y = \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \sum_{i=1}^{n} (y_2 - y_1)_i x_i \cdot \Delta x = \int_a^b (y_2 - y_1)x\,dx.$$

We have seen that $A = \int_a^b (y_2 - y_1)dx$; therefore

$$(4) \qquad \begin{cases} \bar{x} = \dfrac{\displaystyle\int_a^b (y_2 - y_1)x\,dx}{\displaystyle\int_a^b (y_2 - y_1)dx}, \\[2ex] \bar{y} = \dfrac{\displaystyle\frac{1}{2}\int_a^b (y_2^2 - y_1^2)dx}{\displaystyle\int_a^b (y_2 - y_1)dx}. \end{cases}$$

These results may also be derived by double integration if we consider the elements $\Delta y \cdot \Delta x$ and the points (x_i, y_j), the centroids of the elements. We have at once

$$(5)\quad \begin{cases} \bar{x} = \dfrac{\displaystyle\int_a^b \int_{y_1}^{y_2} x\, dy\, dx}{\displaystyle\int_a^b \int_{y_1}^{y_2} dy\, dx}, \\[2ex] \bar{y} = \dfrac{\displaystyle\int_a^b \int_{y_1}^{y_2} y\, dy\, dx}{\displaystyle\int_a^b \int_{y_1}^{y_2} dy\, dx}. \end{cases}$$

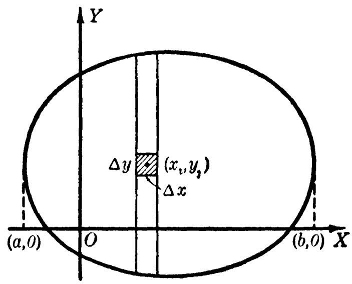


FIG. 208

Evidently the first integration of formulas (5) gives the single integration formulas (4).

If the area is more easily found by means of a horizontal element, formulas corresponding to (4) can be derived. It is then usually advisable to interchange the order of integration in using the formulas (5).

EXAMPLE

Find the centroid of the area bounded by the parabola $x = y^2$ and $y = 2 - x$.

SOLUTION. Using a horizontal element of length l_i and width Δy, we have

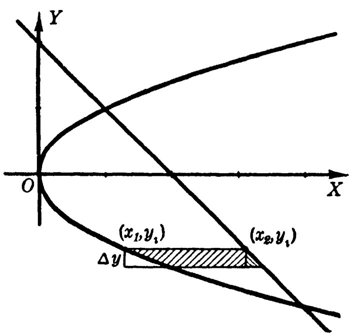


FIG. 209

$$(\Delta M_x)_i = (x_2 - x_1)_i y_i \cdot \Delta y + e_i \cdot \Delta y,$$

and

$$(\Delta M_y)_i = \tfrac{1}{2}(x_2^2 - x_1^2)_i \cdot \Delta y + e_i \cdot \Delta y.$$

Therefore

$$\bar{x} = \frac{\frac{1}{2}\displaystyle\int_{-2}^{1} [(2 - y)^2 - y^4] dy}{\displaystyle\int_{-2}^{1} (2 - y - y^2) dy}$$

$$= \frac{\frac{1}{2}\left(4y - 2y^2 + \frac{y^3}{3} - \frac{y^5}{5}\right)\Big]_{-2}^{1}}{\left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)\Big]_{-2}^{1}} = \frac{8}{5}.$$

and

$$\bar{y} = \frac{\displaystyle\int_{-2}^{1} (2 - y - y^2) y\, dy}{\displaystyle\int_{-2}^{1} (2 - y - y^2) dy} = \frac{\left(y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right)\Big]_{-2}^{1}}{\left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)\Big]_{-2}^{1}} = -\frac{1}{2}.$$

The coordinates of the centroid may be set up as double integrals as follows:

$$\bar{x} = \frac{\int_{-2}^{1}\int_{y^2}^{2-y} x\,dx\,dy}{\int_{-2}^{1}\int_{y^2}^{2-y} dx\,dy}, \qquad \bar{y} = \frac{\int_{-2}^{1}\int_{y^2}^{2-y} y\,dx\,dy}{\int_{-2}^{1}\int_{y^2}^{2-y} dx\,dy}.$$

181. Centroids Using Polar Coordinates. The element $r_i \cdot \Delta r \cdot \Delta\theta$ with centroid (r_i, θ_j) has this centroid $r_i \sin\theta_j$ and $r_i \cos\theta_j$ distant from the polar axis and the radius vector $\theta = \pi/2$, respectively. Hence if $\bar{r}\cos\bar{\theta}$ and $\bar{r}\sin\bar{\theta}$ represent these distances for the centroid of an area, we have

$$\bar{r}\cos\bar{\theta} = \frac{\int_{\alpha}^{\beta}\int_{r_1}^{r_2} r^2\cos\theta\,dr\,d\theta}{\int_{\alpha}^{\beta}\int_{r_1}^{r_2} r\,dr\,d\theta}, \quad \bar{r}\sin\bar{\theta} = \frac{\int_{\alpha}^{\beta}\int_{r_1}^{r_2} r^2\sin\theta\,dr\,d\theta}{\int_{\alpha}^{\beta}\int_{r_1}^{r_2} r\,dr\,d\theta}.$$

EXAMPLE

Find the centroid of the part of the cardioid between $\theta = 0$ and $\theta = \pi/2$.

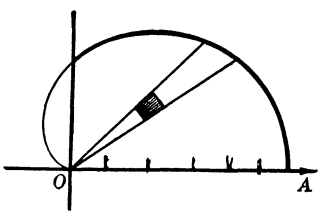


FIG. 210

SOLUTION. Using the formulas given above, we have

$$\bar{r}\cos\bar{\theta} = \frac{\int_0^{\pi/2}\int_0^{a(1+\cos\theta)} r^2\cos\theta\,dr\,d\theta}{\int_0^{\pi/2}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta}$$

$$= \frac{\frac{a^3}{3}\int_0^{\pi/2}(1+\cos\theta)^3\cos\theta\,d\theta}{\frac{a^2}{2}\int_0^{\pi/2}(1+\cos\theta)^2 d\theta}$$

$$= \frac{\frac{a^3}{3}\left[\frac{15\,\theta}{8} + 4\sin\theta - \sin^3\theta + \sin 2\,\theta + \frac{\sin 4\,\theta}{32}\right]_0^{\pi/2}}{\frac{a^2}{2}\left[\frac{3\,\theta}{2} + 2\sin\theta + \frac{\sin 2\,\theta}{4}\right]_0^{\pi/2}} = \frac{16 + 5\,\pi}{16 + 6\,\pi}a.$$

$$\bar{r}\sin\bar{\theta} = \frac{\int_0^{\pi/2}\int_0^{a(1+\cos\theta)} r^2\sin\theta\,dr\,d\theta}{\int_0^{\pi/2}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta} = \frac{\frac{a^3}{3}\int_0^{\pi/2}(1+\cos\theta)^3\sin\theta\,d\theta}{\frac{8+3\,\pi}{8}a^2}$$

$$= \frac{10\,a}{8 + 3\,\pi}.$$

182. Centroid of an Arc of a Plane Curve. The centroid of an arc may be found by considering the moments of elements Δs_i with centroids (x_i, y_i), about the axes. As in previous articles, we have

$$\bar{x} = \frac{\lim\limits_{\substack{n \to \infty \\ \Delta s_i \to 0}} \sum\limits_{i=1}^{n} x_i \Delta s_i}{\lim\limits_{\substack{n \to \infty \\ \Delta s_i \to 0}} \sum\limits_{i=1}^{n} \Delta s_i} = \frac{\int x\,ds}{\int ds},$$

and

$$\bar{y} = \frac{\lim\limits_{\substack{n \to \infty \\ \Delta s_i \to 0}} \sum\limits_{i=1}^{n} y_i \Delta s_i}{\lim\limits_{\substack{n \to \infty \\ \Delta s_i \to 0}} \sum\limits_{i=1}^{n} \Delta s_i} = \frac{\int y\,ds}{\int ds}.$$

EXAMPLE

Find the centroid of a quadrant of the circular arc of radius a.

SOLUTIONS. (*a*) Assume the circle given by $x^2 + y^2 = a^2$. Then

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \frac{x^2}{y^2}}\,dx = \frac{a}{y}\,dx.$$

Therefore

$$\bar{x} = \frac{\int_0^a x\frac{a}{y}\,dx}{\int_0^a \frac{a}{y}\,dx} = \frac{-a\int_a^0 dy}{a\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}}$$

$$= \frac{y\Big]_0^a}{\sin^{-1}\frac{x}{a}\Big]_0^a} = \frac{2a}{\pi}.$$

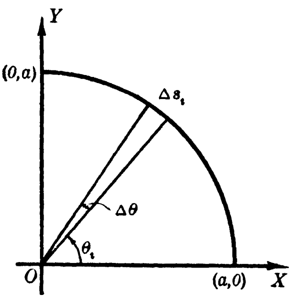


FIG. 211

From symmetry, $\bar{y} = \bar{x} = 2a/\pi$.

(*b*) Suppose we use the fact that the circle can be represented by the parametric equations $x = a \cos\theta$, $y = a\sin\theta$, and that $ds = a\,d\theta$; then we may write

$$\bar{y} = \frac{\int_{\theta=0}^{\theta=\pi/2} y\,ds}{\int_{\theta=0}^{\theta=\pi/2} ds} = \frac{\int_0^{\pi/2} a^2\sin\theta\,d\theta}{\int_0^{\pi/2} a\,d\theta} = \frac{a^2(-\cos\theta)\Big]_0^{\pi/2}}{a\theta\Big]_0^{\pi/2}} = \frac{2a}{\pi},$$

and similarly for $\bar{x}$.

If possible, axes of symmetry should be used in locating centroids.

183. Centroid of a Volume. Considering the moments of elements of a volume about the coordinate planes, we have in a similar manner $(\Delta M_{yz})_{ijl} = \Delta V_{ijl} \cdot x_i$, etc. The same type of reasoning as in previous articles gives

$$\bar{x} = \frac{\lim \sum\sum\sum x_i \cdot \Delta z \cdot \Delta y \cdot \Delta x}{\lim \sum\sum\sum \Delta z \cdot \Delta y \cdot \Delta x} = \frac{\int\int\int x\,dz\,dy\,dx}{\int\int\int dz\,dy\,dx},$$

and similarly

$$\bar{y} = \frac{\int\int\int y\,dz\,dy\,dx}{\int\int\int dz\,dy\,dx}, \qquad \bar{z} = \frac{\int\int\int z\,dz\,dy\,dx}{\int\int\int dz\,dy\,dx}.$$

EXAMPLE

Find the centroid of the solid bounded above by the plane $z = 4$ and below by the paraboloid $z = x^2 + y^2$.

Solutions. (*a*) From symmetry we have $\bar{x} = 0$, $\bar{y} = 0$. Then

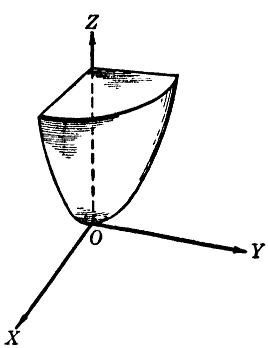


Fig. 212

$$\bar{z} = \frac{4\int_0^2\int_0^{\sqrt{4-x^2}}\int_{x^2+y^2}^4 z\,dz\,dy\,dx}{4\int_0^2\int_0^{\sqrt{4-x^2}}\int_{x^2+y^2}^4 dz\,dy\,dx}$$

$$= \frac{\int_0^2\int_0^{\sqrt{4-x^2}} \left.\frac{z^2}{2}\right]_{x^2+y^2}^4 dy\,dx}{\int_0^2\int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2)dy\,dx}$$

$$= \frac{\frac{1}{2}\int_0^2 \left.\left(16\,y - x^4y - \frac{2\,x^2y^3}{3} - \frac{y^5}{5}\right)\right]_0^{\sqrt{4-x^2}} dx}{\int_0^2 \left.\left(4\,y - x^2y - \frac{y^3}{3}\right)\right]_0^{\sqrt{4-x^2}} dx}$$

$$= \frac{\frac{4}{15}\int_0^2 (6 + x^2)(4 - x^2)^{3/2}dx}{\frac{2}{3}\int_0^2 (4 - x^2)^{3/2}dx}.$$

These integrals can be evaluated by means of the substitution $x = 2 \sin \theta$, but the length of this solution leads us to suggest another.

(*b*) In cylindrical coordinates, the equation of the paraboloid is $z = r^2$. Then

$$\bar{z} = \frac{4\int_0^{\pi/2}\int_0^2\int_{r^2}^4 zr\,dz\,dr\,d\theta}{4\int_0^{\pi/2}\int_0^2\int_{r^2}^4 r\,dz\,dr\,d\theta} = \frac{\frac{1}{2}\int_0^{\pi/2}\int_0^2 (16 - r^4)r\,dr\,d\theta}{\int_0^{\pi/2}\int_0^2 (4 - r^2)r\,dr\,d\theta}$$

$$= \frac{\frac{1}{2}\int_0^{\pi/2}\left(8\,r^2 - \frac{r^6}{6}\right)\Big]_0^2 d\theta}{\int_0^{\pi/2}\left(2\,r^2 - \frac{r^4}{4}\right)\Big]_0^2 d\theta} = \frac{\frac{32}{3}\int_0^{\pi/2} d\theta}{4\int_0^{\pi/2} d\theta} = \frac{8}{3}.$$

(*c*) Still another method is to consider the solid as generated by revolving $z = x^2$ about the z axis. Then, taking disc-shaped elements

$$\Delta V_i = \pi x_i^2 \cdot \Delta z + e_i \cdot \Delta z,$$

we have

$$\bar{z} = \frac{\pi\int_0^4 x^2 z\,dz}{\pi\int_0^4 x^2 dz} = \frac{\int_0^4 z^2 dz}{\int_0^4 z\,dz} = \frac{\frac{z^3}{3}\Big]_0^4}{\frac{z^2}{2}\Big]_0^4} = \frac{8}{3}.$$

The student should consider the several possible ways of solving such problems before attempting to apply any one method.

184. Centroids of Non-homogeneous Bodies. The results of the preceding articles enable us to find the centroids of *thin plates, wires,* and *solids of uniform density.* However, if the material under consideration has its density variable according to some law, the elements ΔA_i, Δs_i, ΔV_i become $\rho\Delta A_i$, $\rho\Delta s_i$, $\rho\Delta V_i$ respectively, where ρ is expressed as a function of one or more of the variables involved. Then

$$\bar{x} = \frac{\int \rho x\,ds}{\int \rho\,ds}, \qquad \bar{y} = \frac{\int \rho y\,ds}{\int \rho\,ds},$$

and similar relations for areas and volumes. These formulas evidently reduce to the relations previously given if ρ is constant.

PROBLEMS

Find the centroid of each of the areas bounded as follows. (Nos. 1–3.)

1. By $y^2 = 4\,x$, and $2\,x - y = 4$. *Ans.* (8/5, 1).

2. A quadrant of a circle.

3. A quadrant of an ellipse. *Ans.* $(4\,a/3\,\pi,\ 4\,b/3\,\pi)$.

4. Prove that the centroid of the area under any parabolic arch is two-fifths the distance from the base to the vertex.

Find the centroid of each of the areas bounded as follows. (Nos. 5–15.)

5. By $y = \cos x$ and the coordinate axes, from $x = 0$ to $x = \pi/2$.
Ans. $[(\pi/2) - 1,\ \pi/8]$.

6. The cardioid $r = a(1 - \cos\theta)$.

7. The limaçon $r = 3 + 2\sin\theta$. *Ans.* $(20/11,\ \pi/2)$.

8. The coordinate axes and the parabola $x^{1/2} + y^{1/2} = a^{1/2}$.

9. A quadrant of the hypocycloid $x = a\cos^3\theta$, $y = a\sin^3\theta$.
Ans. $\bar{x} = \bar{y} = 256\,a/315\,\pi$.

10. The cissoid $y^2 = x^3/(2\,a - x)$ and its asymptote $x = 2\,a$.

11. One loop of $r = a\cos 2\,\theta$. *Ans.* $[(128\,a\sqrt{2})/105\,\pi,\ 0]$.

12. One arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ and the x axis.

13. Between $y = x\log x$ and the x axis. *Ans.* $(4/9,\ -8/27)$.

14. Between $y = xe^{-x}$ and its asymptote.

15. The area in the loop of the curve $y^2 = 4\,x^2 - x^3$ above the x axis.
Ans. $(16/7,\ 5/4)$.

Find the centroid of each of the following arcs of curves. (Nos. 16–18.)

16. The semicircular arc of $y = \sqrt{a^2 - x^2}$.

17. The arc of the cycloid in Problem 12, from $\theta = 0$ to $\theta = \pi$.
Ans. $(4\,a/3,\ 4\,a/3)$.

18. The arc of the cardioid $r = a(1 + \cos\theta)$ above the polar axis.

Find the distance of the centroid of the following surfaces of revolution from the plane of the base. (Nos. 19–21.)

19. The lateral surface of a cone of altitude h, with a radius of base a units.
Ans. $h/3$ units.

20. A hemisphere of radius a units.

21. The paraboloid 2 ft. high, with a radius of base 4 ft.
Ans. $\dfrac{50 - 12\sqrt{2}}{35} = 0.944$ ft.

Find the centroid of each of the following solids. (Nos. 22–23.)

22. A cone of altitude h, and radius of base a units.

23. A hemisphere of radius a units. *Ans.* $3\,a/8$ units from center.

24. Prove that the centroid of a paraboloid of revolution of altitude h is $h/3$ units from the base.

Find the centroid of the following solids. (Nos. 25–29.)

25. Formed by rotating the area included by $x^2 = 4y$, $x = 4$, and $y = 0$ about $y = 0$. *Ans.* $\bar{x} = 10/3$ units.

26. Formed by rotating $y = \sin x$ about $y = 0$ from $x = 0$ to $x = \pi/2$.

27. Formed by rotating one arch of $y = \cos x$ about $x = 0$.
Ans. $\bar{y} = (\pi + 2)/16$ units.

28. Formed by rotating about the y axis the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the first quadrant.

29. Included between the sphere $z^2 + r^2 = 12$ and the paraboloid $z = r^2$.
Ans. $\bar{z} = 2.16$ units.

30. Prove that the wedge above the xy plane cut from the cylinder $x^2 + y^2 = a^2$ by $y = z$ and $z = 0$ has for its centroid $\bar{y} = 2\bar{z} = 3\pi a/16$.

185. Centroids of Composite Areas. If an area is made up of several parts which are separated, we need nothing new to find the centroid of the whole. The moment of the whole is the sum of the moments of the several parts, and the formulas

$$M_x = A \cdot \bar{y}, \qquad M_y = A \cdot \bar{x}$$

still hold. The only difference from the earlier use is that now

$$A = A_1 + A_2 + A_3 + \cdots, \qquad M_x = M_x' + M_x'' + M_x''' + \cdots,$$

where $M_x' = A_1\bar{y}_1$, $M_x'' = A_2\bar{y}_2$, $M_x''' = A_3\bar{y}_3$, $\cdots$. Hence we have

$$(A_1 + A_2 + A_3 + \cdots)\bar{y} = A_1\bar{y}_1 + A_2\bar{y}_2 + A_3\bar{y}_3 + \cdots,$$

or

$$(1) \qquad \bar{y} = \frac{\sum_{i=1}^{n} A_i\bar{y}_i}{\sum_{i=1}^{n} A_i}.$$

where A_i and $A_i\bar{y}_i$ for each i may be found by either method for single areas. Similarly,

$$(2) \qquad \bar{x} = \frac{\sum_{i=1}^{n} A_i\bar{x}_i}{\sum_{i=1}^{n} A_i}.$$

EXAMPLE

Find the centroid of the area of a square of side s after a circle of radius r has been cut out of it.

SOLUTION. Take the coordinate axes so that the center of the square is at the origin and the center of the circle on the x axis at $(a, 0)$. The area of the square may be considered as composed of the circle and that part of the square outside the circle. The centroid of the square is $(0, 0)$ and that of the circle is $(a, 0)$; their areas are s^2 and πr^2, respectively. The other area is $s^2 - \pi r^2$; assuming its centroid to be at $(\bar{x}_2, \bar{y}_2)$ and using formulas (1) and (2) above, we have

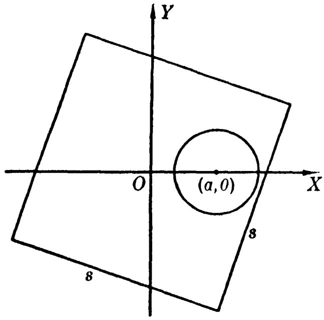


FIG. 213

$$0 = \frac{\pi r^2 \cdot a + (s^2 - \pi r^2)\bar{x}_2}{\pi r^2 + (s^2 - \pi r^2)},$$

whence

$$\bar{x}_2 = \frac{\pi r^2 a}{\pi r^2 - s^2}.$$

To get $\bar{y}_2$, we have

$$0 = \frac{\pi r^2 \cdot 0 + (s^2 - \pi r^2)\bar{y}_2}{\pi r^2 + (s^2 - \pi r^2)},$$

or

$$\bar{y}_2 = 0.$$

We observe that any pair of variables $(\bar{x}_i, \bar{y}_i)$ of the formulas of this article may be found if the others are known.

PROBLEMS

1. Find the centroid of that part of an equilateral triangle remaining after a circle has been cut out of it.

Ans. $[(4\pi r^2 a)/(4\pi r^2 - s^2\sqrt{3}), 0]$, if $(0, 0)$ and $(a, 0)$ are centroids of the triangle and circle, respectively.

2. A square is cut from an ellipse. What is the centroid of the part of the ellipse remaining?

3. An ellipse is cut from a circle. Find the centroid of the remaining area.

Ans. $[abc/(ab - r^2), 0]$.

4. A right circular cylinder has a right circular cone removed. What is the centroid of that part of the cylinder remaining?

5. A sphere is removed from a right circular cone. What is the centroid of the remaining part of the cone?

Ans. $[4r^3a/(4r^3 - R^2H), 0, 0]$, where r refers to sphere, R and H refer to cone, $(a, 0, 0)$ is centroid of sphere and $(0, 0, 0)$ is the centroid of cone.

6. Cut a sphere from a regular tetrahedron and find the centroid of the remainder.

7. A right circular cone is removed from a sphere. What is the centroid of the remainder? *Ans.* $[cr^2h/(4\,a^3 - r^2h),\ 0,\ 0]$.

186. Attraction. The law of gravitation is as follows: *Any two particles attract each other with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them.* Two particles of masses m_1 and m_2, respectively, separated by a distance x exert upon each other a force F given by the formula

$$F = K\frac{m_1 m_2}{x^2}, \tag{1}$$

where K is the ***gravitational constant*** of proportionality.

We are now in a position to calculate the attraction between bodies of known masses.

EXAMPLE

Calculate the attraction between a thin straight wire of mass m per unit of length and a particle of mass M at a point P not on the wire.

SOLUTION. Suppose that the particle of mass M is at the point P on the y axis and that the wire lies along the x axis from A to B. Divide the wire into elements Δx of mass $m\Delta x$. Then the approximate force between the particle and any element is $KmM\cdot\Delta x/r_i^2$. The components of this force which are perpendicular and parallel to the wire are

$$\frac{KmM\cdot\Delta x}{r_i^2}\cos\theta_i \quad\text{and}\quad \frac{KmM\cdot\Delta x}{r_i^2}\sin\theta_i,$$

respectively. Therefore the sums of such elements of force are approximate values for the two forces perpendicular and parallel to the wire. That is, an approximation for the force perpendicular to the wire is

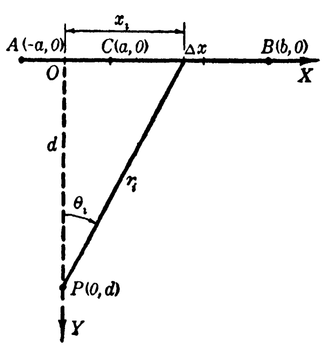


FIG. 214

$$\sum_{i=1}^{n}\frac{KmM\cdot\Delta x\cos\theta_i}{r_i^2},$$

and hence the component of the attraction in that direction is given by

$$A = \lim_{\substack{n\to\infty\\ \Delta x\to 0}}\sum_{i=1}^{n}\frac{KmM\cdot\Delta x\cdot\cos\theta_i}{r_i^2} = KmM\int_{-a}^{b}\frac{\cos\theta\,dx}{r^2}. \tag{2}$$

Similarly, the component parallel to the line is

$$B = KmM \int_{-a}^{b} \frac{\sin\theta\, dx}{r^2}. \tag{3}$$

The values of A and B are readily found since, from Fig. 214, $r^2 = d^2 + x^2$, $\cos\theta = d/\sqrt{d^2 + x^2}$, and $\sin\theta = x/\sqrt{d^2 + x^2}$.

Of course, there is no parallel component if P is on the perpendicular bisector of the wire, and if P is on the wire at O the parallel component is due to the segment $C(a, 0)$ to $B(b, 0)$. Hence relation (3) reduces to

$$B = KmM \int_{a}^{b} \frac{dx}{x^2} = KmM\left(-\frac{1}{x}\right)\Big]_a^b = KmM\,\frac{b-a}{ab}. \tag{4}$$

PROBLEMS

1. Find the attraction of a quadrant of a circular wire on a particle of mass m at its center. *Ans.* $2\,KmM\sqrt{2}/\pi a^2$.

2. Find the attraction on a particle of mass m in the line of a wire of length L whose density varies as the square of the distance from the end nearer the particle.

3. Find the attraction of a circular plate on a particle of mass m on the line through its center perpendicular to the plane of the plate.
Ans. $2\,KmM(1 - d/\sqrt{d^2 + a^2})/a^2$.

4. Find the attraction of a quadrant of a circular wire on a particle of mass m which lies on the circumference of which the wire is a part, and which is opposite the mid-point of the wire.

5. Find the attraction of a square sheet of density ρ on a particle on the line perpendicular to the square at its center.
Ans. $KmM[\pi - 4\sin^{-1}(d/\sqrt{2\,a^2 + 2\,d^2})]/2\,a^2$.

6. Find the attraction of a hemispherical shell on a particle at the center.

7. Find the attraction of a right circular cylinder on a particle at a distance d along its axis from one base.
Ans. $2\,KmM[h + \sqrt{d^2 + a^2} - \sqrt{(d + h)^2 + a^2}]/a^2h$.

8. Find the attraction of a right circular cylindrical shell on a particle at the center of one base.

ADDITIONAL PROBLEMS

1. Find the force on a circular valve of radius 2 ft. placed vertically with its center 20 ft. below the surface of the water. *Ans.* $80\,\pi w$ lbs.

2. Find the force on the shaded area shown in Fig. 215. The vertical parabolic segment is submerged to the top of the shaded area.

3. Find the force on a square flood gate 8 ft. on a side if one side is in the surface, by finding the force on each of the parts into which it is separated by a diagonal.

Ans. $2\frac{2}{3}$ tons, $5\frac{1}{3}$ tons.

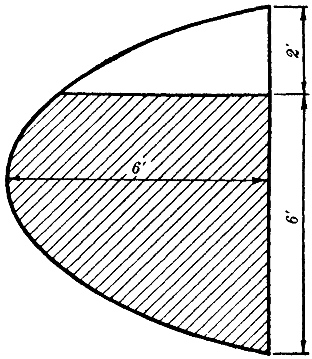


Fig. 215

4. Find the force on the face of the part of an ellipse in the first quadrant if $a = 4$ ft., $b = 2$ ft., and the upper end of the vertical minor axis is 1 ft. below the surface.

5. Find the force on the lower half of the ellipse in Problem 4.

Ans. $4\,w(9\,\pi + 8)/3$ lbs.

6. A plate is in the shape of a parabolic segment, altitude 5 ft., base 6 ft., plane vertical, axis horizontal, and 4 ft. below the surface. Find the force on one face of (a) the complete segment, (b) the part below the axis, (c) the strip from 4 to 5 ft. below the surface.

7. A cross-section of a channel may be represented by a segment of $y = ax^4$. If the water is 6 ft. deep and 10 ft. across the top, find the force on the dam across the channel.

Ans. $128\,w$ lbs.

8. A flood gate is a trapezoid with top base 16 ft., bottom base 10 ft., and the altitude 4 ft. The top is 4 ft. below the surface. What is the force on the gate?

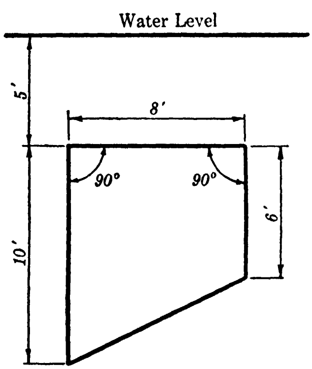


Fig. 216

9. Find the force on the plane area shown in Fig. 216 if it is vertical.

Ans. $1744\,w/3$ lbs.

10. A triangular plate has a side horizontal, 6 ft. long and 2 ft. below the surface. Its altitude is 3 ft. and the vertex is down. Show that the force on one face is independent of the shape of the triangle.

11. A vessel is made by revolving a parabolic segment of altitude 6 ft., and base 4 ft. about its axis. Find the work required to lower the level of the water from 5 ft. to 3 ft. by pumping to 8 ft. above the top.

Ans. $476\,\pi w/9$ ft. lbs.

12. A horizontal cylindrical tank of length 10 ft. and radius 6 ft. is half full of a liquid weighing 50 lbs./cu. ft. Find the work necessary to deliver the liquid to a point 2 ft. above the tank.

13. Three cisterns are each filled with a liquid. One has the shape of a hemisphere, another the shape of a paraboloid of revolution, and the third the shape of a right circular cone. Each has a radius of a, and the same capacity. Compare the units of work required to pump each cistern empty.
Ans. 27 : 32 : 36.

14. A full semi-elliptical reservoir of revolution is 5 ft. deep and 6 ft. across the top. How much work is required to empty it at the top?

15. A hemispherical cistern is full of water. Two men are to pump it out, each doing half of the work. How deep is the water when the first man has finished his work? *Ans.* $r(1 - \sqrt{(2 - \sqrt{2})/2})$ units.

16. A circular disc of uniform thickness weighs 100 lbs., and its radius is 10 inches. Find the work it can do in coming to rest if it spins on an axis through its center and perpendicular to its plane with a rim velocity of 20 feet per second.

17. A thin circular disc of radius a spins in oil about its center with a rim velocity of v. If the oil exerts a resistance on each small element of the disc proportional to the product of the area of the element and its velocity, find the total retarding force on the disc. *Ans.* $4\,k\pi v a^2/3$ units.

18. A slender bar of length L lies along the positive side of the x axis with the nearer end at a distance d from the origin. If the origin is a center of attraction such that each particle of the bar is attracted by a force inversely proportional to the square of its distance from the origin, find the total attraction.

19. The moment of inertia of a plane area about an axis in its plane is equal to its moment of inertia about a parallel axis through its centroid added to the product of the area by the square of the distance between the axes. To show this, take $(\bar{x}, \bar{y})$ of the area at the origin.

20. Prove Problem 19 for a solid of revolution and an axis parallel to its axis of revolution.

21. Show that the force on a plane surface submerged in a vertical position in a fluid is equal to the area of the surface multiplied by the pressure at its centroid.

22. Find the centroid of one-half of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

23. Find the centroid of the area in the first quadrant bounded by $y = \sin x$, $y = \cos x$, and $x = 0$.
Ans. $[(\pi - 2\sqrt{2})/(4 - 2\sqrt{2}),\ 1/(4\sqrt{2} - 4)]$.

24. Find the centroid of each of the areas given below.

(*a*) $\begin{cases} x^2 = 4 - y, \\ x^2 = 4 - 4\,y. \end{cases}$ (*b*) $\begin{cases} y^2 = 16\,x, \\ y = x^2 - 2\,x. \end{cases}$ (*c*) $\begin{cases} 4\,x = y^3, \\ y = x - x^2 + 4. \end{cases}$

25. Find the centroid of a loop of the lemniscate $r^2 = a^2 \cos 2\,\theta$.
Ans. $(\pi a\sqrt{2}/8,\ 0)$.

26. The circles $(x \pm h)^2 + y^2 = a^2$ for $h < a$ form two crescents. Prove that the centroids of these crescents are at $[\pm \pi h/(\pi - 2\theta + \sin 2\theta), 0]$ where $\theta = \cos^{-1}(h/a)$.

27. Revolve the area under $y = \log x$ from $x = 1$ to $x = 3$ about the x axis and find the volume and centroid of the volume generated.

Ans. $1.03\,\pi$ cubic units; $\bar{x} = 2.41$ units.

28. Revolve the area under $y = \sin^2 2x$ from $x = 0$ to $x = \pi/2$ about the x axis and find $\bar{x}$ for the solid by integration.

29. Show that the area generated by revolving an arc of a plane curve about a line in its plane not cutting the arc is the length of the arc multiplied by the circumference of the circle described by the centroid of the arc. (A THEOREM OF PAPPUS.)

30. Show that the volume formed by revolving a plane area around a line in its plane not cutting the area is equal to the area multiplied by the circumference of the circle described by its centroid. (PAPPUS.)

31. (*a*) Use Problem 29 to find the surface of a torus.

Ans. $4\,\pi^2 ab$ sq. units.

(*b*) Use Problem 30 to find the volume of a torus.

Ans. $2\,\pi^2 a^2 b$ cu. units.

32. A square has a side $2a$. Find the centroid of the figure obtained by adding to the square a semicircle having a side of the square as a diameter. Find the centroid if such a semicircle is cut out.

33. Given the parabola $x^2 = 2py$ and any line $y = mx + b$ meeting the parabola in the points A and B. Through C, the midpoint of AB, draw a line parallel to the axis of the parabola meeting the curve at D. Show that the centroid of the area $ACBD$ lies on the line CD.

CHAPTER XVI

INFINITE SERIES WITH CONSTANT TERMS

187. Infinite Series. An indicated sum of n terms formed according to some law is called a ***series*** of n terms. If the number of terms increases indefinitely, the series is called an ***infinite series.***

The geometric series

(1) $$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

is an illustration of a series of n terms. We know that the sum of the first n terms of this series is

(2) $$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

The sum of other series of a finite number of terms may often be expressed conveniently, but the sum of an infinite series will be defined as follows:

(3) $$S = \lim_{n \to \infty} S_n,$$

where S_n is the sum of the first n terms of the series. If this limit does not exist, we say that the series has no sum.

If we consider the geometric series as an infinite series, we write it

(4) $$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots,$$

and then its sum is

(5) $$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(\frac{a}{1 - r} - \frac{ar^n}{1 - r} \right).$$

This limit is $a/(1 - r)$ if $|r| < 1$, as $\lim\limits_{n \to \infty} ar^n = 0$ if $|r| < 1$. But if $|r| > 1$, the quantity ar^n increases indefinitely with n and the series has no sum unless a is zero. Also, if $|r| = 1$, the series has no sum unless a is zero.

188. Convergence and Divergence. If the limit of the sum of the first n terms of a series as n increases indefinitely exists, the series is said to be ***convergent;*** if the limit does not exist the series is said to be ***divergent.*** Hence the geometric series given above is convergent if $|r| < 1$, and divergent if $|r| \geqq 1$ and $a \neq 0$.

Convergent series are of greater importance than divergent ones in elementary applications. For this reason, although we cannot always find $\lim\limits_{n\to\infty} S_n$, it is necessary to know whether such a limit exists. Some methods of determining the existence of this limit will now be given.

189. Comparison Test for Convergence or Divergence. To prove the validity of this test we need the following theorem on limits, which we shall not prove.

THEOREM. *If S_n is a variable quantity which increases (decreases) steadily as n increases but never becomes greater (less) than some fixed finite number S as $n \to \infty$, then S_n approaches a limit L which is not greater (less) than S.* Hence if the sum of the first n terms of an infinite series of *positive terms* is always less than a definite number, whatever n may be, this sum has a limit and the series is convergent.

Since each term of any series is a finite number and since the sum of any finite number of such terms is a finite number, it is evident that the *convergence or divergence of a series is not affected by discarding or adding a finite number of terms.* Now we may state the comparison test as follows:

A series S_2 of positive terms is convergent if after some term each of its terms is less than or equal to the corresponding term of a series S_1 of positive terms which is known to be convergent. Likewise, *S_2 is divergent if after some term each term is equal to or greater than the corresponding term of a series S_1 known to be divergent.*

PROOF. Let

$$a_1 + a_2 + a_3 + \cdots \tag{1}$$

be a positive term series known to be convergent. Suppose

$$u_1 + u_2 + u_3 + \cdots \tag{2}$$

is a positive term series such that $u_k \leqq a_k$.

If A_n and U_n are the sums of the first n terms of (1) and (2), respectively, and if $\lim\limits_{n \to \infty} A_n = A$, it is evident that

$$U_n \leqq A_n < A. \tag{3}$$

Since the series (2) is made up of positive terms, U_n increases as n increases; but, by (3), it is always less than A. This proves that U_n has a limit, by the theorem stated above, and hence series (2) is convergent.

The proof for divergence is obvious.

To apply the test, we must use a convergent series whose terms are ***greater than*** or ***equal to the corresponding terms*** of the series being examined, or a divergent series whose terms are ***less than*** or ***equal to those*** of the unknown series. Evidently, to show that the terms of the unknown series are greater than the corresponding terms of a convergent series, or less than those of a divergent series, is to show nothing about the unknown series.

EXAMPLE

Test by comparison the series

$$\frac{5}{4} + \frac{5}{6} + \frac{5}{10} + \frac{5}{18} + \cdots + \frac{5}{2^n + 2} + \cdots.$$

TEST. Since 5 is a factor of each term, we consider the series

$$\frac{1}{4} + \frac{1}{6} + \frac{1}{10} + \frac{1}{18} + \cdots + \frac{1}{2^n + 2} + \cdots.$$

Comparing the n-th term of this series with the n-th term of the convergent geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots,$$

we assert that

$$\frac{1}{2^n + 2} \leqq \frac{1}{2^n},$$

for, if we clear of fractions, we have

$$2^n \leqq 2^n + 2 \quad \text{or} \quad 0 < 2,$$

which is true for all values of n. Therefore the series is convergent.

The student should notice that the n-th terms of the two series are compared and those only.

190. Cauchy's Ratio Test. *If in a series of **positive terms** the ratio of the $(n + 1)$-st term to the n-th term has a limit L as $n \to \infty$ the series is convergent for $L < 1$, divergent for $L > 1$. The test fails for $L = 1$.*

PROOF. Let the series to be tested be

$$(1) \qquad u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots.$$

By hypothesis, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$. This means that corresponding to any preassigned positive number ϵ, no matter how small, there exists a value of the index, say k, such that the inequality

$$(2) \qquad L - \epsilon < \frac{u_{n+1}}{u_n} < L + \epsilon$$

is satisfied for all values of $n \geqq k$.

CASE I. $L < 1$.

Since L is less than 1, we can find a number r which lies between L and 1, so that $L < r < 1$. If we choose $(r - L)/2$ as the ϵ to be used in (2), we have, from the right side of the inequality,

$$(3) \qquad \frac{u_{n+1}}{u_n} < \frac{L}{2} + \frac{r}{2} < r \quad \text{for all } n \geqq k.$$

From (3), by using $n = k, k + 1, k + 2, \cdots, k + p, \cdots$, we find

$$\begin{aligned}
u_{k+1} &< ru_k \\
u_{k+2} &< ru_{k+1} < r^2u_k \\
u_{k+3} &< ru_{k+2} < r^3u_k \\
&\cdots\cdots\cdots \\
u_{k+p} &< ru_{k+p-1} < r^pu_k \\
&\cdots\cdots\cdots
\end{aligned}$$

Hence each term, after the first, of the series

$$(4) \qquad u_k + u_{k+1} + u_{k+2} + \cdots + u_{k+p} + \cdots$$

is less than the corresponding term of the series

$$(5) \qquad u_k + ru_k + r^2u_k + \cdots + r^pu_k + \cdots.$$

But the series (5) is a geometric series with ratio $r < 1$; hence it converges (as shown in §§ 187 and 188).

Therefore, by applying the theorem of § 189, we see that the series (4) also converges. Consequently, the series (1) converges.

CASE II. $L > 1$.

Since L is greater than 1, we can find a number R which lies between 1 and L, so that $1 < R < L$. If we choose $(L - R)/2$ as the ϵ to be used in (2), we have, from the left side of the inequality,

$$\frac{u_{n+1}}{u_n} > \frac{L}{2} + \frac{R}{2} > R \quad \text{for all } n \geqq k. \tag{6}$$

From (6), by using $n = k, k + 1, k + 2, \cdots, k + p, \cdots$, we find

$$\begin{aligned}
u_{k+1} &> Ru_k \\
u_{k+2} &> Ru_{k+1} > R^2u_k \\
u_{k+3} &> Ru_{k+2} > R^3u_k \\
&\cdots\cdots\cdots \\
u_{k+p} &> Ru_{k+p-1} > R^pu_k \\
&\cdots\cdots\cdots
\end{aligned}$$

Hence each term, after the first, of the series

$$u_k + u_{k+1} + u_{k+2} + \cdots + u_{k+p} + \cdots \tag{7}$$

is greater than the corresponding term of the series

$$u_k + Ru_k + R^2u_k + \cdots + R^pu_k + \cdots. \tag{8}$$

But the series (8) is a geometric series with ratio $R > 1$; hence it diverges (as shown in §§ 187 and 188).

Therefore, by applying the Theorem of § 189, we see that the series (7) also diverges. Consequently, the series (1) diverges.

CASE III. $L = 1$.

Neither of the sets of inequalities given above hold in this case and the test fails to determine whether the series is convergent or divergent. The examples below show this to be true.

(*a*) Try the ratio test on the ***harmonic series***

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots.$$

Test. The n-th term is $1/n$, the $(n+1)$-st term is $1/(n+1)$; therefore

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1.$$

This series will be shown to be divergent in the next article.

(*b*) Use the ratio test on

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}+\cdots.$$

Test. The ratio $u_{n+1}/u_n = n(n+1)/(n+1)(n+2) = n/(n+2)$. Whence

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\frac{n}{n+2} = \lim_{n\to\infty}\frac{1}{1+\frac{2}{n}} = 1.$$

This series is convergent, as will be shown later.

EXAMPLES

1. Use the ratio test on

$$\frac{3}{2}+\frac{4}{3}\cdot\frac{1}{3}+\frac{5}{4}\cdot\frac{1}{3^2}+\cdots+\frac{n+2}{n+1}\cdot\frac{1}{3^{n-1}}+\cdots.$$

Test. The $(n+1)$-st term is $(n+3)/[(n+2)\cdot 3^n]$; therefore

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left[\frac{n+3}{(n+2)3^n}\Big/\frac{n+2}{(n+1)3^{n-1}}\right] = \lim_{n\to\infty}\frac{3^{n-1}(n+1)(n+3)}{3^n(n+2)^2}$$

$$= \lim_{n\to\infty}\frac{n^2+4n+3}{3(n^2+4n+4)} = \frac{1}{3}\lim_{n\to\infty}\frac{1+\frac{4}{n}+\frac{3}{n^2}}{1+\frac{4}{n}+\frac{4}{n^2}} = \frac{1}{3},$$

and hence the series is convergent.

2. Use the ratio test on

$$\frac{1}{2}+\frac{2!}{2^2}+\frac{3!}{2^3}+\cdots+\frac{n!}{2^n}+\cdots.$$

Test. The ratio

$$\frac{u_{n+1}}{u_n} = \frac{2^n(n+1)!}{2^{n+1}\cdot n!} = \frac{n+1}{2}.$$

Therefore

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{1}{2}\lim_{n\to\infty}(n+1) = \infty,$$

and the series is divergent.

191. The Integral Test. If we have *a series of positive terms*

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots,$$

where u_n is expressed as some $f(n)$, and if the function $f(x)$ is defined for all $x \geqq a > 0$ and if $f(x)$ decreases steadily as x increases from a to ∞, the series converges if $\int_a^\infty f(x)dx$ is finite and diverges if the integral is infinite.

We illustrate the test by means of some examples.

EXAMPLES

1. Use the integral test on the harmonic series

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots.$$

TEST. From the form of the n-th term we have the function $f(x) = 1/x$, where x replaces n in that term.

The graph of the function from $x = 1$ to $x = b$ is shown in Fig. 217. The area under this curve as $b \to \infty$ is represented by

$$\lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty}(\log x)\Big]_1^b$$

$$= \lim_{b\to\infty}(\log b) = \infty.$$

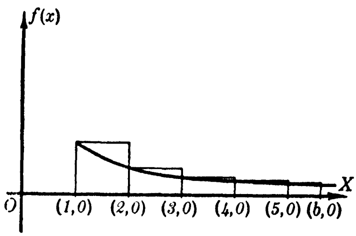


FIG. 217

Since this area is infinite, construct rectangles with unit bases and with altitudes equal to the ordinates of the curve at $x = 1, 2, 3, \cdots, n$. These *rectangles circumscribe* the curve as shown in the figure. Also the area of each rectangle is equal in numerical value to the corresponding term of the series being tested. As these combined areas are greater than that under the curve from $x = 1$ to $x = n + 1$, we have

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_1^{n+1}\frac{dx}{x} = \log(n+1).$$

Therefore

$$S = \lim_{n\to\infty} S_n \geqq \lim_{n\to\infty}\int_1^{n+1}\frac{dx}{x} = \infty,$$

and the series is divergent.

2. Apply the integral test to the series whose n-th term is $1/n^k$ where $k \neq 1$.

TEST. CASE I. $k < 1$.

The curve to be used is the graph of the function $f(x) = 1/x^k$. The area under the curve from $x = 1$ to $x = \infty$ is

$$\lim_{b \to \infty} \int_1^b \frac{dx}{x^k} = \lim_{b \to \infty} \left(\frac{x^{1-k}}{1-k}\right)\Bigg]_1^b$$

$$= \frac{1}{1-k} \lim_{b \to \infty} (b^{1-k} - 1) = \infty.$$

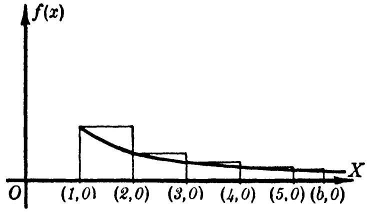


FIG. 218

Since this is infinite, we *circumscribe rectangles* as in the previous example. These show that

$$S_n = \frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k} > \frac{1}{1-k}(n+1)^{1-k}.$$

Hence

$$S = \lim_{n \to \infty} S_n > \frac{1}{1-k} \lim_{n \to \infty} (n+1)^{1-k} = \infty,$$

and the series is divergent.

CASE II. $k > 1$.

The area under the graph of

$$f(x) = \frac{1}{x^k}$$

from $x = 1$ to $x = \infty$ is

$$\lim_{b \to \infty} \int_1^b \frac{dx}{x^k} = \lim_{b \to \infty} \left(\frac{x^{1-k}}{1-k}\right)\Bigg]_1^b = \lim_{b \to \infty} \frac{b^{1-k} - 1}{1-k} = \lim_{b \to \infty} \frac{1 - \dfrac{1}{b^{k-1}}}{k-1} = \frac{1}{k-1},$$

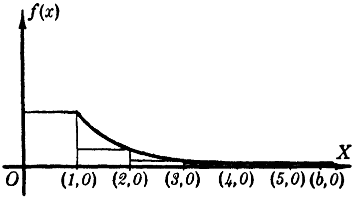


FIG. 219

a finite number. *Since the area is finite,* construct rectangles with unit bases and altitudes equal to the ordinates of the curve at $x = 1, 2, 3, \cdots, n + 1$, which are *inscribed under the curve,* as shown in Fig. 219.

The area of each rectangle is numerically equal to the corresponding term of the series being tested. Therefore the area under the curve from $x = 1$ to $x = n$ is greater than the combined areas of the first n rectangles except for the first rectangle, which is not under that part of the curve. Hence we have

$$S_n - 1 = \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k} < \int_1^n \frac{dx}{x^k} = \frac{1 - \dfrac{1}{n^{k-1}}}{k-1},$$

and therefore

$$S - 1 = \lim_{n\to\infty} (S_n - 1) < \lim_{n\to\infty} \int_1^n \frac{dx}{x^k} = \frac{1}{k-1},$$

or

$$S < 1 + \frac{1}{k-1},$$

and the series is convergent.

These examples give a group of series of the form

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k} + \cdots,$$

which, together with any finite multiples of themselves, are convergent if $k > 1$ and divergent if $k \leqq 1$. These may be used for comparison with unknown series.

The integral test may be readily applied when the *series is a positive term series* with an n-th term which is *integrable.*

If the $f(x)$ derived from the n-th term becomes infinite after $x = 1$, we merely use that part of the graph of the function which is to the right of any vertical asymptotes. This will make the first rectangle constructed correspond to some term after the first in the series and hence the test will compare the sum of a series with an integral where some of the first terms of the original series have been discarded.

192. Alternating Series. This type of series has its alternate terms negative.

Theorem. *If after some term each term of an alternating series is less than or equal to the preceding, that is,* $u_{k+1} \leqq u_k$, *and if* $\lim_{n\to\infty} u_n = 0$, *the series is convergent.*

Proof. Let

$$S_n = u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1}u_n,$$

where each $u_k \geqq u_{k+1}$. Now consider n ***even*** and write S_n in the two forms below:

(1) $\quad S_n = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{n-1} - u_n),$

(2) $\quad S_n = u_1 - (u_2 - u_3) - \cdots - (u_{n-2} - u_{n-1}) - u_n.$

Since $u_{k+1} \leqq u_k$, each parenthesis is positive or zero, and so, from (1), S_n is positive and increases with n or remains unchanged. The relation (2) shows that S_n is less than u_1. Hence S_n has a limit.

Considering an ***odd number*** of terms

$$S_{n+1} = S_n + (-1)^n u_{n+1},$$

we find that this sum has the same limit as S_n since

$$\lim_{n\to\infty} S_{n+1} = \lim_{n\to\infty} S_n + \lim_{n\to\infty} (-1)^n u_{n+1},$$

but by hypothesis $\lim\limits_{n\to\infty} (-1)^n u_{n+1} = 0$. Therefore the series is convergent.

It is evident that *the error in using n terms of a convergent alternating series to represent its sum is less than the* $(n+1)$*-st term.* This is because

$$(3) \qquad S - S_n = u_{n+1} - u_{n+2} + u_{n+3} - \cdots,$$

and this when written as (2) above shows that the difference $S - S_n$ is less than u_{n+1}.

193. Absolute Convergence. A series is said to be ***absolutely convergent*** if the series derived from it by making all terms positive is convergent. If a series is convergent but becomes divergent when all terms are made positive, it is said to be ***conditionally convergent.*** Thus

$$\frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \cdots$$

is absolutely convergent but

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is conditionally convergent.

THEOREM. *If a series is absolutely convergent, it is a convergent series.*

PROOF. Let

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

By hypothesis

$$\bar{S} = \lim_{n \to \infty} \bar{S}_n = \lim_{n \to \infty} (|a_1| + |a_2| + |a_3| + \cdots + |a_n|) = C,$$

a finite number. But

$$S_n = (|a_i| + |a_j| + \cdots + |a_s|) - (|a_k| + |a_l| + \cdots + |a_t|),$$

where the first parenthesis contains only positive terms and the second only negative terms of the series. But

$$\lim_{n \to \infty} (|a_1| + |a_2| + |a_3| + \cdots + |a_n|) = C,$$

and hence

$$\lim_{n \to \infty} (|a_i| + |a_j| + \cdots) \leqq \lim_{n \to \infty} (|a_1| + |a_2| + \cdots + |a_n|) = C,$$

and also

$$\lim_{n \to \infty} (|a_k| + |a_l| + \cdots) \leqq \lim_{n \to \infty} (|a_1| + |a_2| + \cdots + |a_n|) = C.$$

Since each of these limits exists, then

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} (|a_i| + |a_j| + \cdots) - \lim_{n \to \infty} (|a_k| + |a_l| + \cdots)$$

exists and the series is convergent, which was to be proved.

This theorem allows us to use the Cauchy ratio test for series some of whose terms are negative. In so doing, we consider the *absolute value* of the ratio of the $(n + 1)$-st term to the n-th term, *i.e.*, $|u_{n+1}/u_n|$.

A very important *necessary* condition for convergence is that $\lim_{n \to \infty} u_n = 0$. This condition is however *not sufficient*, as is exemplified by the harmonic series.

EXAMPLE

Test the series

$$3 - \frac{3}{2!} + \frac{3}{3!} - \frac{3}{4!} + \cdots$$

for convergence.

TEST. This is an alternating series and hence, if we can show that the $(n + 1)$-st term is less than the n-th and that the limit of the n-th term as n increases indefinitely is zero, it is convergent.

The numerical value of the n-th term is $|u_n| = 3/n!$, also $|u_{n+1}| = 3/(n+1)!$. Therefore set

$$\frac{3}{(n+1)!} < \frac{3}{n!}$$

whence

$$3\,n! < 3(n+1)!,$$

or

$$1 < n + 1,$$

which is true for all n.

Also $\lim\limits_{n\to\infty} (3/n!) = 0$. Therefore the series is convergent.

PROBLEMS

Test the following series for convergence or divergence. Several tests may be applied by the student in most of the problems. Test the alternating series for absolute convergence. (Nos. 1–11.)

1. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2\,n} + \cdots.$ *Ans.* Divergent.

2. $\frac{1}{2} + \frac{2^4}{2^5 + 1} + \frac{3^4}{3^5 + 1} + \cdots.$

3. $2 + \frac{3}{1\cdot 2} + \frac{4}{2\cdot 3} + \frac{5}{3\cdot 4} + \cdots.$ *Ans.* Divergent.

4. $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \cdots.$

5. $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots.$ *Ans.* Convergent.

6. $4 + \frac{8}{3} + \frac{12}{3^2} + \frac{16}{3^3} + \cdots.$

7. $\frac{1}{2} + \frac{1}{3} + \frac{3}{2\cdot 3^2} + \frac{2}{3^3} + \cdots + \frac{n}{2\cdot 3^{n-1}} + \cdots.$ *Ans.* Convergent.

8. $\frac{1}{2} + \frac{1}{9} + \frac{1}{28} + \cdots + \frac{1}{n^3 + 1} + \cdots.$

9. $\frac{1}{2} + \frac{2^2}{2^3 + 1} + \frac{3^2}{3^3 + 1} + \cdots.$ *Ans.* Divergent.

10. $\frac{1}{1 + 2} + \frac{8}{16 + 2} + \frac{27}{81 + 2} + \cdots.$

11. $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots.$ *Ans.* Divergent.

Use the integral test on each of the following series. (Nos. 12–22.)

12. $1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots.$

13. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots.$ *Ans.* Divergent.

14. $1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots.$

15. $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots.$ *Ans.* Convergent.

16. $\frac{4}{2^2 - 1} + \frac{6}{3^2 - 1} + \cdots.$

17. $\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \cdots.$ *Ans.* Convergent.

18. $1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \cdots.$

19. $1 + \frac{1}{2^{3/2} - 1} + \frac{1}{3^{3/2} - 1} + \frac{1}{4^{3/2} - 1} + \cdots.$ *Ans.* Convergent.

20. $\sin 5^\circ + \frac{1}{2} \sin 5^\circ + \frac{1}{3} \sin 5^\circ + \cdots.$

21. $e^{-1} + e^{-2} + e^{-3} + \cdots.$ *Ans.* Convergent.

22. $2\,e^{-2} + 3\,e^{-3} + 4\,e^{-4} + \cdots.$

Test each of the following series for convergence and for absolute convergence.

23. $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots.$ *Ans.* Conditionally conv.

24. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$

25. $\frac{1}{1^3 + 1} - \frac{2^2}{2^3 + 1} + \frac{3^2}{3^3 + 1} - \cdots.$ *Ans.* Conditionally conv.

26. $\frac{1}{2 \cdot 3} - \frac{2}{2^2 \cdot 6} + \frac{3}{2^3 \cdot 11} - \frac{4}{2^4 \cdot 18} + \cdots.$

27. $\frac{1}{3} - \frac{3}{5} + \frac{5}{7} - \frac{7}{9} + \cdots.$ *Ans.* Divergent.

28. $\frac{2}{2} - \frac{4}{3} + \frac{6}{4} - \frac{8}{5} + \cdots.$

29. $\frac{1}{2} - \frac{2}{2 \cdot 3^2} + \frac{1}{2 \cdot 5^2} - \cdots.$ *Ans.* Absolutely conv.

30. $\frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \cdots.$

31. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots.$ *Ans.* Absolutely conv.

32. $\frac{1}{1-2} - \frac{4}{8-2} + \frac{9}{27-2} - \cdots.$

33. $\frac{3}{1 \cdot 2} - \frac{4}{2 \cdot 3} + \frac{5}{3 \cdot 4} - \frac{6}{4 \cdot 5} + \cdots.$ *Ans.* Conditionally conv.

CHAPTER XVII

POWER SERIES AND SOME APPLICATIONS

194. Power Series. Series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

where a_0, a_1, a_2, a_3, $\cdots$ are constants, are called ***power series.***

A power series may be convergent for only some values of the variable or for all values. We state without proof a theorem which we shall need.

ABEL'S THEOREM. *If a power series converges for $x = c$, it converges for any x such that $|x| < c$.*

All of the values of the variable for which a power series is convergent constitute the ***interval of convergence*** of the series. The ratio test may be used to find the interval of convergence in most cases. The values of the variable for which the limit of $|u_{n+1}/u_n| = 1$ are called the ***end points*** of the interval; to determine whether the series is convergent for them or not, we must use some other constant-term series test.

EXAMPLES

1. Find the interval of convergence of

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

SOLUTION. For this series $|u_n| = |x^n/n|$ and therefore

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \left(\frac{x^{n+1}}{n+1}\right) \Big/ \left(\frac{x^n}{n}\right) \right| = \lim_{n \to \infty} |x| \cdot \frac{n}{n+1} = |x|.$$

Whence the series converges for values of x such that $|x| < 1$, that is, for $-1 < x < 1$. The series diverges for $|x| > 1$ or if $x > 1$ and $x < -1$. For $|x| = 1$ the test fails and we must use the values $x = \pm 1$ in the series in order to get constant term series to test for convergence by some other test. If we let $x = -1$ in the series we have

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots,$$

which is divergent, as it is the harmonic series except for sign. Then for $x = 1$ the series becomes

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

which is convergent since

$$\lim_{n \to \infty} \frac{1}{n} = 0 \quad \text{and} \quad \frac{1}{n+1} < \frac{1}{n} \text{ for all } n.$$

The values ± 1 for x are the end points of the interval of convergence of this series and we have shown that $x = 1$ belongs to the interval but $x = -1$ does not. Therefore the interval of convergence of the series is

$$-1 < x \leqq 1.$$

2. Find the interval of convergence of

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots.$$

Solution. As $|u_n| = |x^{2n-1}/(2n-1)!|$ we have

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{x^{2n+1}}{(2n+1)!} \Big/ \frac{x^{2n-1}}{(2n-1)!} \right| = x^2 \lim_{n \to \infty} \frac{1}{2n(2n+1)} = 0$$

for all finite values of x. Therefore the series converges for all finite values of x. Such an interval is written as follows:

$$-\infty < x < \infty.$$

3. Find the interval of convergence of

$$\frac{x-5}{5} + \frac{1}{2} \cdot \frac{(x-5)^2}{5^2} + \frac{1}{3} \cdot \frac{(x-5)^3}{5^3} + \cdots.$$

Solution. Here $|u_n| = |(x-5)^n/n \cdot 5^n|$ and therefore

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{(x-5)^{n+1}}{(n+1) \cdot 5^{n+1}} \cdot \frac{n \cdot 5^n}{(x-5)^n} \right|$$

$$= \left| \frac{x-5}{5} \right| \lim_{n \to \infty} \frac{n}{n+1} = \left| \frac{x-5}{5} \right|.$$

The series is convergent then if $|(x-5)/5| < 1$. That is, for

$$-1 < \frac{x-5}{5} < 1,$$

or

$$-5 < x - 5 < 5,$$

or

$$0 < x < 10,$$

and divergent for $x < 0$ and $x > 10$. Now set $x = 0$ in the series. We have

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots,$$

which we have shown in Example 1 to be convergent by the alternating series test. For $x = 10$, the series becomes

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots,$$

which is divergent. Therefore the interval of convergence is

$$0 \leqq x < 10.$$

The interval of convergence is of great importance in that such a series does not represent some $f(x)$ outside the interval of convergence. Even in the interval of convergence the series should not be used for $f(x)$ unless the remainder after n terms approaches zero rapidly.

PROBLEMS

Find the interval of convergence of each of the following series. Be sure to test the end points of each interval.

1. $\frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots.$ *Ans.* $-1 \leqq x \leqq 1.$

2. $\frac{1}{2} + \frac{x}{2\cdot 3} + \frac{x^2}{3\cdot 4} + \cdots.$

3. $1 + \frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \cdots.$ *Ans.* $-1 \leqq x \leqq 1.$

4. $\frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{6} - \frac{x^4}{8} + \cdots.$

5. $1 - \frac{x}{2} + \frac{x^2}{3\cdot 2^2} - \frac{x^3}{5\cdot 2^3} + \frac{x^4}{7\cdot 2^4} - \cdots.$ *Ans.* $-2 < x \leqq 2.$

6. $1 + \frac{x}{3} + \frac{x^2}{3^2\cdot 2} + \frac{x^3}{3^3\cdot 3} + \cdots.$

7. $\frac{x}{5} + \frac{x^2}{5^2\sqrt{2}} + \frac{x^3}{5^3\sqrt{3}} + \cdots.$ *Ans.* $-5 \leqq x < 5.$

8. $1 + \frac{x}{2} + \frac{x^2}{2^2\cdot 2} + \frac{x^3}{2^3\cdot 3} + \cdots.$

9. $\frac{x-3}{3} - \frac{1}{2}\cdot\frac{(x-3)^2}{3^2} + \frac{1}{3}\cdot\frac{(x-3)^3}{3^3} - \cdots.$ *Ans.* $0 < x \leqq 6.$

10. $1 + \frac{x-2}{3} + \frac{(x-2)^2}{5} + \frac{(x-2)^3}{7} + \cdots.$

11. $\frac{x+1}{4} + \frac{2(x+1)^2}{4^2} + \frac{3(x+1)^3}{4^3} + \cdots.$ *Ans.* $-5 < x < 3.$

12. $1 + \frac{x+3}{\sqrt{2}} + \frac{(x+3)^2}{\sqrt{3}} + \frac{(x+3)^3}{2} + \cdots.$

13. $\frac{x+2}{3} - \frac{(x+2)^2}{3^2 \cdot 2^{1/3}} + \frac{(x+2)^3}{3^3 \cdot 3^{1/3}} - \cdots.$ *Ans.* $-5 < x \leqq 1.$

14. $1 + \frac{(x+2)^2}{2^2} + \frac{(x+2)^4}{3^2} + \frac{(x+2)^6}{4^2} + \cdots.$

15. $1 + \frac{x-1}{1+2^2} + \frac{(x-1)^2}{1+3^2} + \cdots.$ *Ans.* $0 \leqq x \leqq 2.$

16. $\frac{x+1}{3} - \frac{(x+1)^2}{3^2\sqrt{2}} + \frac{(x+1)^3}{3^3\sqrt{3}} - \cdots.$

17. $\frac{x+2}{3 \cdot 2} + \frac{(x+2)^2}{3^2 \cdot 5} + \frac{(x+2)^3}{3^3 \cdot 10} + \cdots.$ *Ans.* $-5 \leqq x \leqq 1.$

18. $\frac{x+k}{4} + \frac{2(x+k)^2}{4^2} + \frac{3(x+k)^3}{4^3} + \cdots.$

19. Is $1 + \frac{2}{3}(x-1) + \frac{4}{6}(x-1)^2 + \frac{8}{9}(x-1)^3 + \cdots$ convergent for $x = 0, -1, \frac{1}{4}, \frac{3}{4}, 5$? *Ans.* Only for $x = 3/4$.

195. Taylor's and Maclaurin's Theorems. Since a power series becomes a convergent constant-term series for any value of the variable within its interval of convergence, its sum is a function of the variable within the interval.

To find a series which represents a given function for some interval is the important problem which is now presented.

Suppose we have a continuous function $f(x)$ with continuous first, second, $\cdots$, and $(n+1)$-th derivatives, denoted by $f'(x)$, $f''(x)$, $\cdots$, $f^{n+1}(x)$, in the interval from $x = a$ to $x = b$, inclusive.

(A) *Using the Extended Theorem of Mean Value.*

The proof of the law of the mean given in § 90 can be restated, for a function $f(x)$ which has a continuous derivative, by setting

$$\frac{f(b) - f(a)}{b - a} = K_1,$$

or

(1a) $$f(b) - f(a) - K_1(b - a) = 0,$$

so that the auxiliary function there used becomes

(2a) $$\phi(x) = f(b) - f(x) - K_1(b - x),$$

which reduces to the left side of (1a) when $x = a$. This leads, as in § 90, to the final formula proved there, which may be written in the form

(3a) $$f(b) = f(a) + f'(x_1)(b - a), \quad a < x_1 < b.$$

Proceeding in a precisely analogous manner, let us write

$$\frac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = K_2,$$

or

(1b) $$f(b) - f(a) - f'(a)(b - a) - K_2(b - a)^2 = 0.$$

If we now examine the auxiliary function

(2b) $$\phi(x) = f(b) - f(x) - f'(x)(b - x) - K_2(b - x)^2,$$

we observe that $\phi(a) = 0$ by (1b), and $\phi(b) = 0$ identically. Hence, by Rolle's theorem (§ 89), we must have $\phi'(x_2) = 0$ for some value x_2 between a and b. But, from (2b),

$$\phi'(x) = 0 - f'(x) + f'(x) - f''(x)(b - x) + 2\,K_2(b - x),$$

whence, since $\phi'(x_2) = 0$, we have

$$-f''(x_2)(b - x_2) + 2\,K_2(b - x_2) = 0,$$

so that $2\,K_2 = f''(x_2)$, since $b - x_2$ can not be zero. Therefore (1b) may be written in the form

$$f(b) = f(a) + f'(a)(b - a) + f''(x_2)\,\frac{(b - a)^2}{2}, \quad a < x_2 < b.$$

In the same manner, if we commence with

(1) $$f(b) - f(a) - f'(a)(b - a) - f''(a)\,\frac{(b - a)^2}{2\,!} - \cdots - f^n(a)\,\frac{(b - a)^n}{n\,!} - K_{n+1}(b - a)^{n+1} = 0,$$

and use the auxiliary function

$$(2)\quad \phi(x) = f(b) - f(x) - f'(x)(b-x) - f''(x)\frac{(b-x)^2}{2!} - \cdots$$
$$- f^n(x)\frac{(b-x)^n}{n!} - K_{n+1}(b-x)^{n+1},$$

we see again that $\phi(a) = 0$ and $\phi(b) = 0$, so that $\phi'(x_{n+1}) = 0$, where x_{n+1} lies between a and b. Proceeding exactly as above, we thus obtain the formula

$$(3)\quad f(b) = f(a) + f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2!} + \cdots$$
$$+ f^n(a)\frac{(b-a)^n}{n!} + f^{n+1}(x_{n+1})\frac{(b-a)^{n+1}}{(n+1)!},$$

where $a < x_{n+1} < b$.

It is evident that the same formula would be obtained by using in the place of b any value of x greater than a. Replacing b by x in (3), we may then write

$$(4)\quad f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \cdots$$
$$+ f^n(a)\frac{(x-a)^n}{n!} + f^{n+1}(x_{n+1})\frac{(x-a)^{n+1}}{(n+1)!},$$

where $a < x_{n+1} < x \leqq b$. The same formula holds also if $b < a$, if all the inequalities throughout are reversed, and if finally $a > x_{n+1} > x \geqq b$.

This is known as ***Taylor's theorem*** and the function $f(x)$ is said to be ***expanded about*** or ***in the neighborhood of $x = a$***.

If we set $a = 0$ in Taylor's theorem, we have

$$(5)\quad f(x) = f(0) + f'(0)\cdot x + f''(0)\cdot\frac{x^2}{2!} + \cdots + f^n(0)\cdot\frac{x^n}{n!}$$
$$+ f^{n+1}(\bar{x})\cdot\frac{x^{n+1}}{(n+1)!}, \quad \text{where} \quad 0 < \bar{x} < x,$$

which is known as ***Maclaurin's theorem*** for $f(x)$, and $f(x)$ is said to be ***expanded in the neighborhood of $x = 0$***.

(B) *Using Successive Integrations.**

We form the expression (§ 150, Problems 103–106)

$$\int_a^x \left[f^{n+1}(y) \frac{(x-y)^n}{n!} \right] dy, \tag{6}$$

and apply integration by parts, setting $u = (x-y)^n/n!$ and $dv = f^{n+1}(y)dy$. Then $du = [-(x-y)^{n-1}/(n-1)!]dy$ and $v = f^n(y)$. Therefore

$$\begin{aligned}\int_a^x \left[f^{n+1}(y) \frac{(x-y)^n}{n!} \right] dy &= \left[f^n(y) \cdot \frac{(x-y)^n}{n!} \right]_a^x + \int_a^x \left[f^n(y) \cdot \frac{(x-y)^{n-1}}{(n-1)!} \right] dy \\ &= -f^n(a) \frac{(x-a)^n}{n!} + \int_a^x \left[f^n(y) \cdot \frac{(x-y)^{n-1}}{(n-1)!} \right] dy.\end{aligned}$$

Another integration by parts gives

$$\begin{aligned}\int_a^x \left[f^{n+1}(y) \frac{(x-y)^n}{n!} \right] dy &= -f^n(a) \frac{(x-a)^n}{n!} - f^{n-1}(a) \frac{(x-a)^{n-1}}{(n-1)!} \\ &\quad + \int_a^x \left[f^{n-1}(y) \frac{(x-y)^{n-2}}{(n-2)!} \right] dy.\end{aligned}$$

Successive integrations give

$$\begin{aligned}\int_a^x \left[f^{n+1}(y) \frac{(x-y)^n}{n!} \right] dy &= -f^n(a) \frac{(x-a)^n}{n!} - f^{n-1}(a) \frac{(x-a)^{n-1}}{(n-1)!} - \cdots \\ &\quad - f''(a) \frac{(x-a)^2}{2!} + \int_a^x f''(y)(x-y)dy \\ &= -f^n(a) \frac{(x-a)^n}{n!} - f^{n-1}(a) \frac{(x-a)^{n-1}}{(n-1)!} - \cdots \\ &\quad - f''(a) \frac{(x-a)^2}{2!} - f'(a)(x-a) + f(x) - f(a);\end{aligned}$$

* This method of proof may be omitted if desired.

whence, by transposition,

$$(7)\quad f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x-a)^2}{2!} + \cdots$$

$$+ f^n(a)\frac{(x-a)^n}{n!} + \int_a^x \left[f^{n+1}(y)\frac{(x-y)^n}{n!}\right] dy.$$

Then $a = 0$ gives

$$(8)\quad f(x) = f(0) + f'(0)\cdot x + f''(0)\cdot\frac{x^2}{2!} + \cdots + f^n(0)\cdot\frac{x^n}{n!}$$

$$+ \int_0^x \left[f^{n+1}(y)\frac{(x-y)^n}{n!}\right] dy.$$

These formulas are Taylor's and Maclaurin's theorems with an integral for the last term.

(C) *Using Undetermined Coefficients.*

We shall here adopt another point of view in the derivation of Maclaurin's series for a function $f(x)$ which possesses derivatives of all orders but complete proofs will not be attempted.

If a power series

$$(9)\quad a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + a_{n+1}x^{n+1} + \cdots$$

converges for any value of x different from zero, as $x = \rho$, it will converge in an interval $-\rho < x < \rho$, where $\rho > 0$ (§ 194).

In more advanced works in mathematics, it is shown that if the power series (9) converges within a certain interval $-\rho < x < \rho$, with $\rho > 0$, to a function $f(x)$, then the power series which is obtained by termwise differentiation will converge within the same interval to the derivative of $f(x)$. By repeated applications of this theorem, we can then see that if a convergent power series is repeatedly differentiated termwise the successive derived series will converge to the successive derivatives of $f(x)$ within the interval $-\rho < x < \rho$.

We shall assume that there is a power series which converges to the function $f(x)$ for all values of x within the interval $-\rho < x < \rho$, with $\rho > 0$. Let

$$(10)\quad f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + a_{n+1}x^{n+1} + \cdots.$$

Our problem is to determine the values of the coefficients a_0, a_1, $a_2, \cdots, a_n, \cdots$. By the aforementioned property of power series, the successive derived series will converge to the successive derivatives of $f(x)$ within the interval $-\rho < x < \rho$. Forming these successive derived series, we have

$$(11)\quad \begin{cases} f'(x) = a_1 + 2\,a_2x + 3\,a_3x^2 + \cdots + na_nx^{n-1} \\ \qquad\qquad + (n+1)a_{n+1}x^n + \cdots, \\ f''(x) = 2\cdot 1\,a_2 + 3\cdot 2\,a_3x + \cdots + n(n-1)\,a_nx^{n-2} \\ \qquad\qquad + (n+1)\cdot n\cdot a_{n+1}x^{n-1} + \cdots, \\ f'''(x) = 3\cdot 2\cdot 1\,a_3 + \cdots + n(n-1)(n-2)\,a_nx^{n-3} \\ \qquad\qquad + (n+1)n(n-1)\,a_{n+1}x^{n-2} + \cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f^{(n)}(x) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\,a_n \\ \qquad\qquad + (n+1)\,n(n-1)(n-2)\cdots \\ \qquad\qquad\quad 4\cdot 3\cdot 2\cdot a_{n+1}x + \cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

Since the series in (10) and (11) converge for $x = 0$ to the values of $f(x)$ and its successive derivatives, respectively, at $x = 0$, we have, by letting $x = 0$:

$$(12)\quad \begin{cases} f(0) = a_0 \\ f'(0) = a_1 \\ f''(0) = 2\,!\,a_2 \\ f'''(0) = 3\,!\,a_3 \\ \cdots\cdots\cdots \\ f^{(n)}(0) = n\,!\,a_n \\ \cdots\cdots\cdots \end{cases}$$

By solving the equations of (12) for the coefficients $a_0, a_1, a_2, \cdots a_n, \cdots$, we have

$$a_0 = f(0),\ a_1 = f'(0),\ a_2 = \frac{f''(0)}{2\,!},\ \cdots,\ a_n = \frac{f^{(n)}(0)}{n\,!},\ \cdots.$$

Therefore we obtain

$$(13)\quad f(x) = f(0) + f'(0)x + \frac{f''(0)}{2\,!}x^2 + \cdots + \frac{f^{(n)}(0)}{n\,!}x^n + \cdots.$$

This is Maclaurin's series for $f(x)$. For a given function, the coefficients of the series can be obtained and the interval of convergence of the resulting power series can be obtained by the methods of § 194. It can be shown that the series obtained actually converges within this interval to the function $f(x)$ provided that the remainder E_{n+1} of § 196 approaches zero as n becomes infinite; but the proof is beyond the scope of this book.

196. Applications of Taylor and Maclaurin Theorems. We say that we compute the values of some functions by using series into which the functions have been expanded. Such series should have their terms decrease rapidly, since we never use the infinite series in computation problems. At best only a few terms are used, and unless the terms decrease rapidly more terms will have to be computed. The most important thing is the value of the ***remainder*** of the series when $(n + 1)$ terms have been used as an approximation. In (4) and (5) of § 195, this remainder is the last term whose value depends on $f^{n+1}(x_{n+1})$ where $a < x_{n+1} < x \leqq b$ (or, if $b < a$, $b \leqq x < x_{n+1} < a$). An upper limit for the value of the remainder after $n + 1$ terms may be found by using the maximum value of $|f^{n+1}(x)|$ in the interval from a to x. Suppose that the largest numerical value of $f^{n+1}(x)$ within the interval of integration is given by $x = c$. Then $|f^{n+1}(c)|$ represents that greatest numerical value. Then we see that the ***remainder*** or ***error***, which we shall denote by E_{n+1}, for the Taylor expansion, satisfies the inequality

$$(1) \qquad |E_{n+1}| \leqq \left| f^{n+1}(c) \frac{(x-a)^{n+1}}{(n+1)!} \right|,$$

where c is the value for which $|f^{n+1}(x)|$ is a maximum for all values of x under consideration; and, for the Maclaurin expansion,

$$(2) \qquad |E_{n+1}| \leqq \left| f^{n+1}(c) \frac{x^{n+1}}{(n+1)!} \right|.$$

Similar formulas can be derived also from (7) and (8) of § 195.

Three important questions arise in using series for numerical calculations, namely:

(a) What error is made in using a given number of terms?

(*b*) How many terms must be used to get a value of the function correct to a given number of decimal places?

(*c*) Within what interval of the variable will a given number of terms give required accuracy?

We illustrate methods of answering these questions by means of special examples.

Since a Taylor series is composed of powers of $(x - a)$, the rapidity of its convergence depends upon the numerical value of $(x - a)$. Hence, in expanding a function, we choose a so that $|x - a|$ is as small as possible and at the same time so that the values of $f(a), f'(a), f''(a)$, etc., are easily computed.

EXAMPLES

1. Find a series suitable for evaluating cos 31°.

SOLUTION. A suitable series means one for which $|x - a|$ is small and $f(a)$, $f'(a)$, etc., are readily computed. Since 30° is near the angle to be used and sin 30° and cos 30° are readily written down, we take $a = \pi/6$. Then

$$f(x) = \cos x, \qquad f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},$$

$$f'(x) = -\sin x, \qquad f'\left(\frac{\pi}{6}\right) = -\frac{1}{2},$$

$$f''(x) = -\cos x, \qquad f''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2},$$

$$f'''(x) = \sin x, \qquad f'''\left(\frac{\pi}{6}\right) = \frac{1}{2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence, by substitution in the formula (4) of § 195, we have

$$\cos x = \frac{\sqrt{3}}{2} - \frac{x - \frac{\pi}{6}}{2} - \frac{\sqrt{3}\left(x - \frac{\pi}{6}\right)^2}{2 \cdot 2\,!} + \frac{\left(x - \frac{\pi}{6}\right)^3}{2 \cdot 3\,!} + \cdots.$$

For $x = 31\,\pi/180$, the first three terms give

$$\cos 31° = \frac{\sqrt{3}}{2} - \frac{\pi}{360} - \frac{\sqrt{3}\left(\frac{\pi}{180}\right)^2}{4} + \cdots$$

$$= 0.866025 - 0.008727 - 0.000132 + \cdots$$

$$= 0.85717 \cdots.$$

This is correct to five decimals, for the error after three terms, E_3, satisfies the inequality

$$|E_3| \leqq \frac{1}{3!}\sin c\left(\frac{\pi}{180}\right)^3 \leqq \frac{1}{3!}\left(\frac{\pi}{180}\right)^3 = 0.00000088.$$

2. How many terms of the expansion of $\log x$ in the neighborhood of $x = 3$ are needed to get log 3.5 correct to four decimal places?

Solution. In relation (1) of this article we have $|E_{n+1}| < 0.00005$, $a = 3$, $x = 3.5$, and n is to be determined. Since

$$f^{n+1}(x) = \frac{d^{n+1}\log x}{dx^{n+1}} = (-1)^n\frac{n!}{x^{n+1}},$$

the maximum value of $|f^{n+1}(x)|$ in the interval $3 \leqq x \leqq 3.5$ is evidently

$$|f^{n+1}(3)| = \left|(-1)^n\cdot\frac{n!}{3^{n+1}}\right| = \frac{n!}{3^{n+1}}.$$

Then we must have

$$|E_{n+1}| \leqq \left[\frac{n!}{3^{n+1}}\right]\left[\frac{(0.5)^{n+1}}{(n+1)!}\right] < 0.00005.$$

The smallest possible value of n satisfying this inequality is found by trial.

In $|E_{n+1}| \leqq (0.5)^{n+1}/[3^{n+1}(n+1)]$ the value $n = 2$ gives $|E_3| \leqq 0.00154$; $n = 3$ gives $|E_4| \leqq 0.00019$; $n = 4$ gives $|E_5| \leqq 0.000026$. Thus $n = 4$ makes the error as small as desired. *Since* $(n + 1)$ *terms precede the term representing the value of the remainder* E_{n+1} *of the series* and we have found $n = 4$, it is necessary to use five terms of the series.

3. What accuracy is given if six terms of the series of Example 2 are used?

Solution. As $n = 5$ if six terms are used, we have

$$|E_6| \leqq \frac{1}{3^6}\cdot\frac{(0.5)^6}{6} = 0.00000357.$$

Therefore six terms will give log 3.5 correct to at least five decimals.

4. Find the interval within which two terms of the series of Example 2 will give an error not exceeding 0.005.

Solution. Since $n + 1 = 2$ and $f(x) = \log x$, we need $f''(x) = -1/x^2$. Therefore

$$|E_2| \leqq \frac{1}{c^2}\cdot\left|\frac{(x-3)^2}{2}\right|,$$

where c lies between 3 and x. To make $|E_2| \leqq 0.005$ we must determine x so that

$$\left|\frac{1}{c^2}\cdot\frac{(x-3)^2}{2}\right| \leqq 0.005.$$

There are two cases to consider.

CASE I. WHEN $x > 3$. In this case $c = 3$, and we have

$$\frac{1}{9} \cdot \frac{(x-3)^2}{2} \leqq 0.005,$$

whence $x \leqq 3.3$.

CASE II. WHEN $x < 3$. The maximum value of $|f''(x)|$ is now $1/x^2$ and $|x - 3|$ is $3 - x$. Therefore

$$\frac{1}{x^2} \cdot \frac{(3-x)^2}{2} \leqq 0.005,$$

whence $x \geqq 2.727$.
Therefore the complete interval is

$$2.727 \leqq x \leqq 3.3.$$

However, we may consider the two cases simultaneously so as to give an ***interval symmetric about*** $x = 3$. To do this we let $|x - 3| = h$ and $c = 3 + e$, where $|e| \leqq h$. Then the inequality used above becomes

$$\frac{1}{(3+e)^2} \cdot \frac{h^2}{2} \leqq 0.005.$$

The maximum value of $1/(3 + e)^2$ is for $e = -h$ as this makes $3 + e$ least, so that our inequality becomes

$$\frac{1}{(3-h)^2} \cdot \frac{h^2}{2} \leqq 0.005,$$

whence $h \leqq 0.2727$. Hence $|x - 3| = 0.2727$ and the symmetric interval is

$$-0.2727 \leqq x - 3 \leqq 0.2727,$$

or

$$2.7273 \leqq x \leqq 3.2727.$$

PROBLEMS

1. Assume $e^x = a + bx + cx^2 + dx^3 + \cdots$, and find $a, b, c, d, \cdots$ by the method of undetermined coefficients.

2. Expand $3y^2 - 7y + 9$ in a series of powers of $y - 3$.

3. Expand $2x^3 - 3x^2 - 8x + 10$ in a series of powers of $x + 2$.

Ans. $-2 + 28(x + 2) - 15(x + 2)^2 + 2(x + 2)^3$.

Expand each of the following functions to four terms, first in a power series of $(x - a)$ (Taylor), and second in a power series of x (Maclaurin). (Nos. 4–8.)

4. $(1 \pm x)^n$.

5. $\sin x$.

$$Ans.\ \begin{cases} \sin a + \cos a \cdot (x - a) - \dfrac{\sin a}{2\,!}(x - a)^2 - \dfrac{\cos a}{3\,!}(x - a)^3 + \cdots, \\ x - x^3/3\,! + x^5/5\,! - x^7/7\,! + \cdots. \end{cases}$$

6. $\cos x$.

7. e^{-x}.

$$Ans.\ \begin{cases} e^{-a}[1 - (x - a) + \dfrac{1}{2\,!}(x - a)^2 - \dfrac{1}{3\,!}(x - a)^3 + \cdots], \\ 1 - x + x^2/2\,! - x^3/3\,! + \cdots. \end{cases}$$

8. $\log(1 \pm x)$.

Expand each of the following functions to three terms using both Taylor's and Maclaurin's series. (Nos. 9–10.)

9. $e^{\sin x}$.

$$Ans.\ \begin{cases} e^{\sin a}[1 + \cos a \cdot (x - a) + (\cos^2 a - \sin a)\dfrac{(x - a)^2}{2\,!} + \cdots], \\ 1 + x + x^2/2\,! \cdots. \end{cases}$$

10. $e^x \cos x$.

11. Write the Maclaurin expansion of e^{-x^2} to four terms and find the interval of convergence. *Ans.* $-\infty < x < \infty$.

12. Expand the following functions in a power series of x.

(*a*) $\sin^{-1} x$.
(*b*) a^x.
(*c*) $\tan^{-1} x$.
(*d*) $\log \cos x$.
(*e*) $\log(\sqrt{x^2 + 1} - x)$.
(*f*) $\tan(x + \pi/4)$.

13. Write a Taylor's series suitable for evaluating sin 29° and approximate the value of sin 29° by using three terms. *Ans.* 0.4849.

14. Same as Problem 13 for cos 62°.

15. Same as Problem 13 for sin 59°. *Ans.* 0.8572.

16. What is an upper limit ($|E_{n+1}|$) of the error in Problem 13?

17. What is an upper limit ($|E_{n+1}|$) of the error in Problem 14? *Ans.* 0.00001.

18. What is an upper limit ($|E_{n+1}|$) of the error in Problem 15?

19. Write several terms of the Maclaurin series for e^x and find an upper limit for error in $e^{0.02}$ if four terms are used. *Ans.* $(10)^{-8} \cdot (0.68)$.

20. How many terms would make the result in Problem 19 correct to four places?

21. Find $|E_{n+1}|$ if $\log x$ is expanded about $x = 3$ and three terms are used to approximate log 3.11. *Ans.* $|E_3| < 0.00002$.

22. How many terms must be used to get the result in Problem 21 correct to four decimals?

23. Derive a series for $\tan^{-1} 0.1$ and evaluate to four decimals.
Ans. 0.0997.

24. Derive a series for $\tan^{-1} 0.2$ and evaluate to four decimals.

25. Derive a series and evaluate $\cos 29°$ to three decimals. *Ans.* 0.875.

26. Derive a series and evaluate $\sin 29° 40'$ to four decimals.

27. Derive a series and evaluate $\cos 46°$ to four decimals. *Ans.* 0.6947.

28. Write the Taylor's series for e^{-x} about $x = 1$ and find the interval of convergence.

29. Use Problem 28 to find an upper limit to the error in $e^{-1.1}$ if three terms are used. *Ans.* $|E_3| < 0.00006$.

30. Write series for $\log(1 - x)$ and evaluate log 0.9 to three decimal places.

31. How many terms of the Maclaurin series for $\log(1 - x)$ are necessary to approximate log 1.05 to four decimal places? *Ans.* 3 terms.

32. Find an upper limit of the error involved in Problem 31.

33. Evaluate log 1.1 to three decimal places. *Ans.* 0.095.

34. Evaluate $\cos 61°$ to four decimal places.

35. What is an upper limit to the error in $\cos 12°$ if four terms of the Maclaurin series is used? *Ans.* $|E_4| < (10)^{-7}$.

36. If $\log 2 = 0.6931$, compute log 2.1 to three decimal places.

37. Within what limits will three terms of the Maclaurin expansion of $\log(1 + x)$ give an error not greater than 0.0001?
Ans. $-0.0717 \leqq x \leqq 0.0669$.

38. Within what limits will three terms of the Maclaurin series for $\cos x$ give an error not greater than 2 units in the fourth decimal place?

39. Within what interval do four terms of the series for $\sin x$ about $\pi/6$ give results correct to five decimals? *Ans.* $0.399 \leqq x \leqq 0.628$.

40. Using (1) of § 196, find the unknown in each of the following cases if $f(x) = \sin x$ and $a = \pi/3$.

(a) $n = 3$, $x = 61° 30'$ expressed in radians, $|E_4| = ?$
(b) $x = 58° 45'$, $|E_{n+1}| \leqq 0.0001$, $n = ?$
(c) $n = 3$, $|E_4| \leqq 0.0005$, $x = ?$

41. The same as Problem 40 for $f(x) = \log x$, $a = 3$.

(a) $n = 3$, $x = 3.2$. *Ans.* $|E_4| < 0.000005$.
(b) $x = 3.2$, $|E_{n+1}| \leqq 0.0001$. *Ans.* $n = 2$.
(c) $n = 3$, $|E_4| \leqq 0.0005$. *Ans.* $2.48 \leqq x \leqq 3.63$.

197. Newton's Method of Approximating the Roots of an Equation. Taylor's series may be used to solve equations approximately. A method due to Newton is illustrated by what follows.

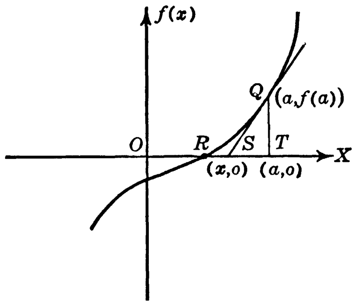


FIG. 220

Suppose $f(x) = 0$ is a given equation. Let Fig. 220 represent the graph of $f(x)$ near a root. By trial we locate a first approximation to the root, such as $x = a$. The slope of the tangent at Q is $f'(a)$, which from the figure is equal to $f(a)/(a - x)$. Therefore x, a second approximation to the root, is given by

$$x = a - \frac{f(a)}{f'(a)}$$

But this is the value of x obtained by replacing $f(x)$ by the first two terms of Taylor's expansion. It may be used as a new value of a to obtain the third approximation, and so on.

EXAMPLE

Find to three decimals the positive root of $x^2 - \cos x = 0$.

SOLUTION. By graphical methods or by use of tables we find that there is a root between 0.8 and 0.9. We now write the equation with all terms in the left-hand member and set

$$f(x) = x^2 - \cos x = 0.$$

If we replace $f(x)$ by the first two terms of Taylor's expansion about 0.8, we have

$$f(x) = f(0.8) + f'(0.8)(x - 0.8) = 0,$$

if we discard higher powers of $x - 0.8$, which itself is less than 0.1. This becomes

$$\begin{aligned} 0.64 - \cos 0.8 + (1.6 + \sin 0.8)(x - 0.8) &= 0, \\ 0.64 - 0.6967 + (1.6 + 0.7174)(x - 0.8) &= 0, \\ x - 0.8 &= 0.024, \\ x &= 0.82, \end{aligned}$$

if we carry the division to two decimal places.

To get a third decimal, we repeat the operation, using the expansion about 0.82. This gives

$$\begin{aligned} f(x) = 0.6724 - \cos 0.82 + (1.64 + \sin 0.82)(x - 0.82) &= 0, \\ x - 0.82 &= 0.0041, \\ x &= 0.8241, \end{aligned}$$

so we conclude that $x = 0.824$ is the solution desired. If the fourth decimal is more than 5, either increase the third decimal by unity or better still repeat the operation using three decimals in the expansion.

PROBLEMS

Solve each of the following equations to three decimals. (Nos. 1–13.)

1. $\cos x - 10\,x = 0$. *Ans.* 0.0995.

2. $e^{x/2} + x = 2$ for its smallest positive root.

3. $e^x + x = 2$ for the root near 0.4. *Ans.* 0.443.

4. $2 \cos x = x - 2$.

5. $e^{2x} - x = 2$. *Ans.* -1.981, 0.448.

6. $x^3 + 4\,x - 7 = 0$.

7. $3 \log x - e^x + 4 = 0$ for the root near 1.7. *Ans.* 1.731.

8. $x^3 = \sin (x + 2)$.

9. $\sin 2\,x + x = 1.4$. *Ans.* 0.529.

10. $2\,x \cos x = x^2 - 1.2$.

11. $x^2 - 2\,x + \log x = 0$. *Ans.* 1.690.

12. $\tan 3\,x = 1 - 3\,x$ for its smallest positive root.

13. $\sin (x/2) - 2\,x + 8 = 0$. *Ans.* 4.403.

14. Find a high and a low point on the curve $y = x \sin x$.

15. Solve the equation $e^{x/2} + \cos 2\,x = x^2 - 2$. *Ans.* -1.293, etc.

16. Solve the equation $e^{-x} + \sin x = 1.235$ for the root between 1.2 and 1.3.

198. Approximations to Definite Integrals. The quantity represented by $\int_a^b f(x)dx$ is well known when the indefinite integral $F(x)$, which has $f(x)$ as its derivative with respect to x, can be found. We recall that the numerical value of $\int_a^b f(x)dx$ is $F(b) - F(a)$. However, $F(b) - F(a)$ cannot be found if we cannot find $F(x)$ and also if we do not know the exact form of $f(x)$, but merely know its values for special values of x. We shall develop two methods of *evaluating* $\int_a^b f(x)dx$ *approximately* which depend upon the ability to find numerical values of $f(x)$ for various values of x in the interval $a \leqq x \leqq b$. These methods are therefore applicable whether or not the form of $F(x)$ is known.

199. The Prismoidal Formula. Let $f(x)$ be expanded in a Taylor's series in the neighborhood of $x = a$. If we represent $f(x)$ by the first four terms of this expansion, we have

$$f(x) = A + B(x - a) + C(x - a)^2 + D(x - a)^3, \tag{1}$$

where A, B, C, and D represent the fractional coefficients of the Taylor's series.

Under this assumption, for $\int_a^b f(x)dx$ *we write*

$$\begin{aligned}(2)\qquad &\int_a^b [A + B(x - a) + C(x - a)^2 + D(x - a)^3]dx \\ &= A(b - a) + \frac{B(b - a)^2}{2} + \frac{C(b - a)^3}{3} + \frac{D(b - a)^4}{4} \\ &= \left(\frac{b - a}{6}\right)\left[6A + 3B(b - a) + 2C(b - a)^2 + \frac{3D(b - a)^3}{2}\right].\end{aligned}$$

From (1) we find the values of $f(x)$ for $x = a$, $(a + b)/2$, and b in terms of A, B, C, D to be

$$(3)\qquad \begin{cases} f(a) = A, \\ f\left(\dfrac{a + b}{2}\right) = A + \dfrac{B(b - a)}{2} + \dfrac{C(b - a)^2}{4} + \dfrac{D(b - a)^3}{8}, \\ f(b) = A + B(b - a) + C(b - a)^2 + D(b - a)^3. \end{cases}$$

If we form the combination

$$f(a) + 4f\left(\frac{a + b}{2}\right) + f(b)$$

from relations (3) we get the quantity in the last bracket in equation (2). Whence

$$(4) \qquad \int_a^b f(x)dx = \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

This is the ***prismoidal formula*** *for approximating to the value of the definite integral* $\int_a^b f(x)dx$. The accuracy of this value depends upon the value of $(x - a)$, as we have discarded powers greater than the third in assuming the value (1) for $f(x)$.

The student will notice that this value of the definite integral depends upon the values of $f(x)$ only for $x = a$, $(a + b)/2$, and b; that is, at the ends and in the center of the interval of integration. For this reason the formula is applicable if $f(x)$ is known or if its value can be found for these special values of the variable of integration.

Evidently the prismoidal formula gives an *exact* result if $f(x)$ is a quadratic function, in fact even if it is a cubic function, as the four terms assumed for $f(x)$ make a cubic function. However, since generally $f(x)$ is not a cubic, it is necessary to consider means of making the approximation better. Due to the fact that the formula will in general give more accurate results when b is near to a, there is an immediate means of improving its accuracy. We may divide the interval from a to b into several smaller intervals and apply the formula to each. As an aid to judicial division, a graph of $f(x)$ should be used. Smaller intervals in which the variation of $f(x)$ is fairly uniform should be chosen for the several applications of the formula. If the figure represents $f(x)$ over the interval a to b, it would be advisable to apply the formula over at least the three intervals a to c, c to d, and d to b for improved accuracy.

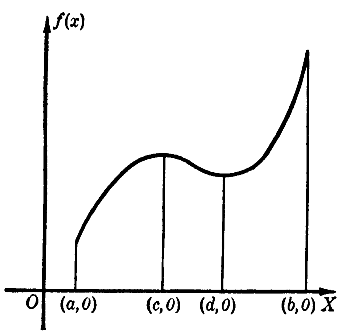


FIG. 221

200. Simpson's Rule. This is merely a special case of the suggestion given above for better accuracy. The interval a to b is divided into smaller intervals of *equal length*, the prismoidal formula is applied to each and the various results are combined

into one formula. Since the smaller intervals are equal, suppose we represent their length by $2\,\Delta x$. Then we have

$$(1)\quad \int_a^b f(x)dx = \int_a^{a+2\,\Delta x} f(x)dx + \int_{a+2\,\Delta x}^{a+4\,\Delta x} f(x)dx$$
$$+ \int_{a+4\,\Delta x}^{a+6\,\Delta x} f(x)dx + \cdots + \int_{b-2\,\Delta x}^{b} f(x)dx.$$

Applying the prismoidal formula to each integral on the right-hand side of (1), we obtain

$$(2)\quad \int_a^b f(x)dx = \frac{2\,\Delta x}{6}[f(a) + 4\,f(a+\Delta x) + f(a+2\,\Delta x)]$$
$$+ \frac{2\,\Delta x}{6}[f(a+2\,\Delta x) + 4\,f(a+3\,\Delta x) + f(a+4\,\Delta x)]$$
$$+ \frac{2\,\Delta x}{6}[f(a+4\,\Delta x) + 4\,f(a+5\,\Delta x) + f(a+6\,\Delta x)]$$
$$+ \cdots + \frac{2\,\Delta x}{6}[f(b-2\,\Delta x) + 4\,f(b-\Delta x) + f(b)],$$

or

$$(3)\quad \int_a^b f(x)dx = \frac{\Delta x}{3}[f(a) + 4\,f(a+\Delta x) + 2\,f(a+2\,\Delta x)$$
$$+ 4\,f(a+3\,\Delta x) + 2\,f(a+4\,\Delta x) + \cdots$$
$$+ 2\,f(b-2\,\Delta x) + 4\,f(b-\Delta x) + f(b)].$$

This formula is called ***Simpson's Rule*** *for approximating a definite integral.*

Again we suggest that the length of the sub-intervals $(2\,\Delta x)$ taken be small for the sake of accuracy. The student is also reminded that the quantity $2\,\Delta x$ must be chosen so as to divide the interval $(b - a)$ exactly.

In applying any of these methods when the form of $f(x)$ is not known and the function is given only by a table of values, *it is necessary to plot the known values of the function and then draw a smooth curve through these points. The values of $f(x)$ at the special points needed are then read from the graph.*

201. Integration of a Series. A third method of approximating the definite integral when integration is impossible or inconvenient is to expand $f(x)$ in a power series and integrate termwise. This method may be used to any desired accuracy if the interval of integration is within the interval of convergence, *the end points of the interval of convergence being excluded.* Rapidly decreasing terms with the *remainder approaching zero* are necessary to make this method useful.

EXAMPLES

1. Find the length of the curve $y = x^2$ from (1, 1) to (2, 4).

SOLUTIONS. The element of arc Δs_i is $\sqrt{1 + 4x_i^2} \cdot \Delta x$ and the desired quantity is represented by $\int_1^2 \sqrt{1 + 4x^2}\, dx$.

(*a*) SOLUTION BY PRISMOIDAL FORMULA.

Since $f(x) = \sqrt{1 + 4x^2}$, and $a = 1$, $(a + b)/2 = 1.5$, $b = 2$, we have $f(a) = \sqrt{5}$, $f[(a + b)/2] = \sqrt{10}$, $f(b) = \sqrt{17}$. Therefore

$$s = \int_1^2 \sqrt{1 + 4x^2}\, dx = \frac{1}{6}(\sqrt{5} + 4\sqrt{10} + \sqrt{17}) = 3.16805.$$

(*b*) SOLUTION BY SIMPSON'S RULE.

Suppose $2\Delta x = 0.5$. Then $f(a) = \sqrt{5}$, $f(a + \Delta x) = (1/2)\sqrt{29}$, $f(a + 2\Delta x) = \sqrt{10}$, $f(b - \Delta x) = (1/2)\sqrt{53}$, and $f(b) = \sqrt{17}$. Therefore

$$s = \frac{0.25}{3}(\sqrt{5} + 2\sqrt{29} + 2\sqrt{10} + 2\sqrt{53} + \sqrt{17}) = 3.16786.$$

The value of this integral correct to six significant figures is 3.16784.

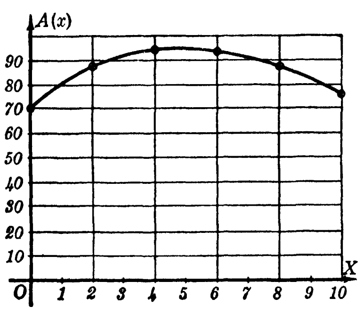


FIG. 222

2. At intervals of 6 ft. the areas in square yards of the cross-sections of a railway cut are 70, 88, 94, 93, 87, 76. How many cubic feet of earth must be removed in making the cut between the two end sections given?

SOLUTIONS. The element of volume is $A(x_i)\Delta x$ where $A(x_i)$ is a cross-section of the cut at a distance x_i yards from one end. Since $A(x_i)$ is only given by a table of values, we must use a graph to approximate the integral representing the desired volume, that is, $\int_0^{10} A(x)dx$.

(*a*) SOLUTION BY PRISMOIDAL FORMULA.

We have given $A(a) = A(0) = 70$, $A(b) = A(10) = 76$; from the graph we get $A[(a+b)/2] = A(5) = 93.5$. Therefore

$$V = \int_0^{10} A(x)dx = \frac{10}{6}[70 + 4(93.5) + 76] = 866.67 \text{ cu. yds.}$$

(*b*) SOLUTION BY SIMPSON'S RULE.

If we take $2\,\Delta x = 2$, the table gives $A(0) = 70$, $A(2) = 88$, $A(4) = 94$, $A(6) = 93$, $A(8) = 87$, $A(10) = 76$. From the graph we estimate that $A(1) = 79$, $A(3) = 91.5$, $A(5) = 93.5$, $A(7) = 89$, $A(9) = 80$. Therefore

$$V = \frac{1}{3}[70 + 4(79) + 2(88) + 4(91.5) + 2(94) + 4(93.5) + 2(93) + 4(89) + 2(87) + 4(80) + 76] = 867.33 \text{ cu. yds.}$$

3. Evaluate $\int_{0.2}^{1} [(\cos x)/x]dx$.

SOLUTION. Since $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots$, and is convergent for all x, and the remainder has the limit zero, we have

$$\int_{0.2}^{1}\left(\frac{\cos x}{x}\right)dx = \int_{0.2}^{1}\left(\frac{1}{x} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots\right)dx$$

$$= \left(\log x - \frac{x^2}{4} + \frac{x^4}{96} - \frac{x}{4320} + \cdots\right)\Big]_{0.2}^{1}$$

$$= 1.37961.$$

PROBLEMS

The answers given are approximations.

1. The width of a sheet of tin is w ft. at a distance x feet from one end. Approximate the area of the sheet defined by:

w ft.	0	0.4	0.8	3.3	3.7	4.1	4
x ft.	0	0.7	1.1	1 8	2.1	3.2	4

Ans. 10.5 sq. ft.

2. A circular pin has the length 20 in. Diameters x in. from one end are given by:

d in.	2	4	5	6	8
x in.	0	5	10	15	20

Write the integral for its approximate volume and evaluate.

3. The diameters of a buoy at x ft. from one end are:

x ft.	0	2	5	6	7	9
D ft.	2	4	6	8	10	9

Approximate its volume. *Ans.* 96 π cu. ft.

4. At 10-ft. intervals, right sections of a vessel are 2.67, 3.89, 5.18, 8.76, 9.13 sq. ft. Approximate its volume.

5. The area bounded by $y = 0$, $x = 2$, $x = 6$ and the curve defined by

x	2.0	2.5	3	4.5	5.5	6
y	10.2	8.3	3.8	5.7	2.1	3.6

is desired. *Ans.* 19.6 sq. units.

6. What volume is formed if the area of Problem 5 is revolved about the x axis?

7. Approximate the length of the parabola $y = 2x^2$ from (1, 2) to (2, 8). Use $\Delta x = 1/4$. Evaluate by integration. *Ans.* 6.09 units; 6.086 units.

8. At 6 ft. intervals the area in square yards of cross-sections of a railway cut are 75, 88, 91, 87, 74. Approximate the volume of earth removed.

9. Approximate the length of $x^2 - y^2 + 9 = 0$ from (0, 3) to (4, 5). Use $\Delta x = 0.5$. *Ans.* 4.553 units.

Approximate the following integrals as suggested or by several methods. (Nos. 10–22.)

10. $\int_0^1 \frac{dx}{1 + 2x^2}$, using $x = 1/4, 1/2, 3/4$ as division points.

11. $\int_0^1 \frac{dx}{3 - x^2}$, using $x = 1/4, 1/2, 3/4$ as division points. *Ans.* 0.380.

12. $\int_0^{1/2} \frac{dx}{1 + x^2}$, by Simpson's rule, using five values of x.

13. $\int_{0.1}^{0.5} \frac{dx}{xe^x}$, using four sub-intervals. *Ans.* 1.276.

14. $\int_{\pi/6}^{\pi/3} \sqrt{2 + \tan x}\, dx$ by prismoidal formula; also using four sub-intervals.

15. $\int_{\pi/6}^{\pi/2} \sqrt{10 \sin x}\, dx$, using four sub-intervals. *Ans.* 0.89 π.

16. $\int_0^{1/2} \frac{\sin \theta}{\theta}\, d\theta$, using five values for θ.

17. $\int_0^{1/2} e^{-x^3}\,dx$, using three terms of a series. *Ans.* 869/1792.

18. $\int_0^{\pi/2} \sqrt{\cos x}\,dx$, using three terms of a series.

19. $\int_0^1 \sqrt[3]{4 + x^2}\,dx$. *Ans.* 1.63.

20. $\int_{0.2}^1 \sqrt{4 - 2x^3}\,dx$.

21. $\int_1^2 \sqrt{1 + x^3}\,dx$. *Ans.* 2.13.

22. $\int_2^4 \sqrt{1 + \log x}\,dx$.

23. Given $y = e^{-x/2}$. (*a*) Approximate its length from $x = 0$ to $x = 2$. (*b*) Revolve the curve about the x axis and calculate approximately the volume of the solid generated. *Ans.* (*a*) 2.10 units, (*b*) 2.72 cu. units.

24. Show that the prismoidal formula gives the exact area between the parabola $y = ax^2 + bx + c$, the x axis, and the ordinates $x = h$ and $x = h + 2\,\Delta x$.

25. Approximate the area of the surface generated by revolving the arc of $x^2 - y^2 = 9$ from $x = 3$ to $x = 5$ around the y axis. *Ans.* 108.28 sq. units.

26. If P is the resultant force of the pressure on the piston of an engine when a weight has been raised to a height of h feet, find the work done in raising the weight (*a*) two feet; (*b*) eight feet, if

h (feet)	0	0 5	1	1.5	2	3	4	5	6	7	8
P (lbs.)	100	110	110	110	100	73	54	44	38	34	30

27. A is the area of the water plane of a vessel at a distance x feet above the keel. If

x (feet)	2	4	6	8	10
A (sq. ft.)	2690	3635	4320	4900	5400

find the displacement of the vessel for a draught of 10 feet.
Ans. 36,537 cu. ft.

28. An oil barrel has the following diameters D inches at the given distances x inches from one end. Approximate its volume.

x inches	0	4	7	11	15.5	20	24	28	31
D inches	21	23	23 5	24	24 25	23.75	23.5	22.5	21.25

29. A body weighing 1610 lbs. was lifted vertically by a rope, there being a damped spring balance to indicate the pulling force of F lbs. When the body

had been lifted x feet from its position of rest, the force was recorded automatically as follows:

x	0	11	20	34	45	55	66	76
F	4010	3915	3763	3532	3366	3208	3100	3007

Find the probable velocity in feet per second at $x = 30$ feet and at $x = 72$ feet. $[mv^2/2 = \int_0^x (F - 1610)dx]$. *Ans.* 51.62 ft./sec., 73.92 ft./sec.

202. Calculation of π. If we expand $\tan^{-1} x$ by Maclaurin's series we have

$$(1) \qquad \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1} \cdot \frac{x^{2n-1}}{2n-1} + \cdots.$$

This is readily shown to be convergent for every value of x in the interval $-1 \leqq x \leqq 1$, and the remainder, which is less than $|x^{2n+1}/(2n+1)|$, approaches zero for all values of x in the interval of convergence. Whence, letting $x = 1, 1/2$, and $1/3$ and using the relation

$$\frac{\pi}{4} = \tan^{-1}(1) = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right),$$

we have, from (1), that

$$\frac{\pi}{4} = \left[\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \frac{\left(\frac{1}{2}\right)^7}{7} + \frac{\left(\frac{1}{2}\right)^9}{9} - \frac{\left(\frac{1}{2}\right)^{11}}{11} + \frac{\left(\frac{1}{2}\right)^{13}}{13} - \cdots\right]$$
$$+ \left[\frac{1}{3} - \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} - \frac{\left(\frac{1}{3}\right)^7}{7} + \frac{\left(\frac{1}{3}\right)^9}{9} - \cdots\right]$$
$$= 0.78540$$

to 5 significant figures. Therefore $\pi = 3.14160$, approximately.

The two series for $\tan^{-1}(1/2)$ and $\tan^{-1}(1/3)$ are used here because they converge more rapidly than does the series for $\tan^{-1}(1)$.

203. Computation of Logarithms. The Maclaurin expansions of log $(1 + x)$ and log $(1 - x)$ allow us to write

$$\begin{aligned}\log\left(\frac{1+x}{1-x}\right) &= \log(1+x) - \log(1-x)\\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)\\ &\qquad - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots\right),\end{aligned}$$

or

$$(1)\quad \log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

Now set $x = 1/(2y + 1)$ and we get

$$\frac{1+x}{1-x} = \frac{1+y}{y},$$

whence from (1)

$$\begin{aligned}\log\left(\frac{1+y}{y}\right) = 2\Big[&\frac{1}{2y+1} + \frac{1}{3(2y+1)^3} + \cdots\\ &+ \frac{1}{(2n-1)(2y+1)^{2n-1}} + \cdots\Big],\end{aligned}$$

or

$$\begin{aligned}(2)\quad \log(y+1) = \log y + 2\Big[&\frac{1}{2y+1} + \frac{1}{3(2y+1)^3} + \cdots\\ &+ \frac{1}{(2n-1)(2y+1)^{2n-1}} + \cdots\Big].\end{aligned}$$

This series is called the ***logarithmic series***, since, given log y, we have a means of computing log $(y + 1)$ at once to any desired accuracy. The region of convergence of (2) is $y > 0$ and the convergence is rapid after $y = 3$. Of course the remainder has the limit zero over the region of convergence.

Logarithms computed by means of series (2) are to the base e; we readily convert them to the base 10 by means of the relation

$$\log_{10} N = \log N/\log 10 = (\log N)/2.30259 = \mathbf{0.43429 \log N}.$$

PROBLEMS

1. Show that the series for $\tan^{-1}x$ about $x = 0$ is not desirable for computing π if $x = 1$.

2. Compute π to four decimals using

$$\tan^{-1}(1) = 2\tan^{-1}(1/2) - \tan^{-1}(1/7).$$

3. Form other trigonometric formulas like that of Problem 2 for $\tan^{-1}(1)$ and compute π to four places.

4. Compute by formula (2) the natural logarithms of the following numbers. Find to as many decimals as requested. (NOTE. Use the laws of logarithms.)

(*a*) 1, (*b*) 2, (*c*) 7, (*d*) 17, (*e*) 24, (*f*) 33, (*g*) 37.

5. Compute by formula (2) the common logarithms of the following numbers. Find to as many decimals as requested.

(*a*) 1/2, (*b*) 1, (*c*) 3, (*d*) 4, (*e*) 25, (*f*) 34, (*g*) 41.

6. Compute the value of π by taking 4 terms of the series for $\sin^{-1}x$ and evaluate for $x = 1/2$.

204. Sin *x* and Cos *x* Expressed as Exponential Functions. The Maclaurin expansions for e^y, $\sin y$, and $\cos y$ are convergent for all real values of y. These expansions, which should be memorized by the student, are

$$(1) \qquad e^y = 1 + y + \frac{y^2}{2\,!} + \frac{y^3}{3\,!} + \frac{y^4}{4\,!} + \cdots,$$

$$(2) \qquad \sin y = y - \frac{y^3}{3\,!} + \frac{y^5}{5\,!} - \frac{y^7}{7\,!} + \cdots,$$

$$(3) \qquad \cos y = 1 - \frac{y^2}{2\,!} + \frac{y^4}{4\,!} - \frac{y^6}{6\,!} + \cdots.$$

Now *if we assume* that these three series represent the functions for $y = ix$, where $i = \sqrt{-1}$ and x is real, we have

$$(4) \quad e^{ix} = \left(1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \cdots\right) + i\left(x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \cdots\right).$$

But the two parentheses are just (3) and (2), and therefore

$$(5) \qquad \boldsymbol{e^{ix} = \cos x + i \sin x.}$$

Similarly we get

$$e^{-ix} = \cos x - i \sin x. \tag{6}$$

These relations give

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \tag{7}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}. \tag{8}$$

Then since $e^{x+iy} = e^x \cdot e^{iy}$, we may use complex numbers for y in (1) and, under the same assumption, find that

$$e^{x+iy} = e^x(\cos y + i \sin y). \tag{9}$$

These relations between trigonometric and exponential expressions are of special importance in certain applications.

ADDITIONAL PROBLEMS

Find the interval of convergence of each of the following series. (Nos. 1–7.)

1. $1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{30} + \cdots.$ *Ans.* $-1 \leqq x \leqq 1.$

2. $\frac{x-4}{2!} + \frac{(x-4)^2}{3!} + \frac{(x-4)^3}{4!} + \cdots.$

3. $\frac{x-2}{2} - \frac{(x-2)^2}{2^2} + \frac{(x-2)^3}{2^3} - \cdots.$ *Ans.* $0 < x < 4.$

4. $(2/3)(x-3) + (3/4)(x-3)^2 + (4/5)(x-3)^3 + \cdots.$

5. $\frac{x-1}{3} - \frac{(x-1)^3}{5} + \frac{(x-1)^5}{7} - \cdots.$ *Ans.* $0 \leqq x \leqq 2.$

6. $\frac{x+2}{3} - \frac{(x+2)^2}{3^2 \cdot 2^{1/2}} + \frac{(x+2)^3}{3^3 \cdot 3^{1/2}} - \cdots.$

7. $\frac{1}{2} + \frac{x-1}{5} + \frac{(x-1)^2}{10} + \cdots.$ *Ans.* $0 \leqq x \leqq 2.$

Solve each of the following equations by Newton's method. (Nos. 8–16.)

8. $x^3 + 2x^2 - 7 = 0.$

9. $x^3 - 2x^2 + \cos x = 0.$ *Ans.* 0.764, etc.

10. $2 \tan 3x = 1 - x.$

11. $\log x + x^2 = 2 \sin x$. *Ans.* 1.291.

12. $xe^{-x} - \cos 2x = 0$.

13. $e^{x/2} = \cos(x/3) + 0.6$. *Ans.* 0.885, etc.

14. $1 - x^2 = \sin(x + 2)$.

15. $e^{x/2} + \cos 2x = x^2 - 2$. *Ans.* -1.294, etc.

16. $\cos \pi x + \sqrt{-x} = 0$. How many roots?

17. Find an upper limit to the error if four terms are used for $\cos x$ about $x = 0$ for $\cos 12°$. *Ans.* 0.00008.

18. The same as Problem 17 for $\sin x$ about $x = \pi/6$ for $\sin 32°$.

19. The same as Problem 17 for $\log(2 + x)$ for three terms to get $\log 2.1$. *Ans.* 0.00004.

20. Derive the series for e^x for x near 2. How many terms are needed to get $e^{1.96}$ to four decimals?

21. How many terms of the Maclaurin series for $\log(2 + x)$ must be taken to get $\log 2.3$ to three decimals? *Ans.* Four terms.

22. Find $\tan^{-1}(1/4)$ to three places by using a suitable series.

23. Evaluate $\log 1.992$ from the series for $\log x$ about $x = 2$. *Ans.* 0.689.

24. Find the interval of convergence of the series in Problem 23 and evaluate $\log 2.01$.

25. Find an upper limit for x which will permit the approximation of $(1 + x)^4$ to within 1 in 1000 by two terms of the Maclaurin expansion. *Ans.* 0.0129.

26. Within what positive interval do three terms of the Maclaurin expansion of e^x give a maximum error of not more than 1 in 1000?

27. Evaluate to three decimals by using sufficient terms of a series each of the following:

(a) $\displaystyle\int_0^1 \sqrt[3]{1 + x^2}\, dx$, (b) $\displaystyle\int_{\pi/12}^{\pi/6} \sin^{1/2} x\, dx$,

(c) $\displaystyle\int_0^{1/2} \sqrt{1 + x^4}\, dx$, (d) $\displaystyle\int_0^{\pi/4} \cos x^2\, dx$.

Ans. (a) 1.09, (b) 0.161, (c) 0.503, (d) 0.756.

28. Find the length of the chord of an arc of radius 200 feet subtending 3° at the center (a) by trigonometric methods, (b) by the approximation formula that the length of the chord equals $r\theta - r\theta^3/24$, where θ is the central angle and r is the radius. Compare the two results. Also derive the approximation formula.

29. Show that the difference between the length of a circular arc and its chord is approximately $r\theta^3/24$. Also show that the error in the approximation cannot exceed $r\theta^5/1920$.

30. A horizontal cylindrical tank, 6 feet long and 3 feet in diameter, has 6 cubic feet of water in it. Find the depth of the water correct to two decimals, by solving an equation by Newton's method.

31. Draw the graphs of $y = x$, $y = x - x^3/3\,!$, $y = x - x^3/3\,! + x^5/5\,!$, with the same coordinate axes and compare them with the graph of $y = \sin x$ from $x = -1$ to $x = 1$.

32. Obtain the binomial theorem by expanding $(a + x)^n$. What is its interval of convergence?

33. The length of the cable of a suspension bridge is given by

$$L = \frac{wl}{2P}\sqrt{\left(\frac{P}{w}\right)^2 + \left(\frac{l}{2}\right)^2} + \frac{P}{w}\log\left[\frac{l}{2} + \sqrt{\left(\frac{P}{w}\right)^2 + \left(\frac{l}{2}\right)^2} \div \frac{P}{w}\right],$$

where l is the length of span, w the weight per unit length horizontally, P the tension in the cable at its lowest point. In general w/P is small. Derive the following formula, which is more convenient for calculation:

$$L = l + \left(\frac{w}{P}\right)^2 \frac{l^3}{24} - \left(\frac{w}{P}\right)^4 \frac{l^5}{768}.$$

34. Can the following functions be expanded in Maclaurin series? Why?
(*a*) $\log x$; (*b*) $\csc x$; (*c*) $\operatorname{ctn}\theta$; (*d*) $\theta \csc^2 \theta$.

35. By means of the Maclaurin expansion of e^x compute
(*a*) $e^{1/2}$; (*b*) $e^{1/3}$; (*c*) $e^{1/5}$; (*d*) e^{-1}.

Ans. 1.6487; 1.3956; 1.2214; 0.36788.

36. Use the binomial theorem to find
(*a*) $(1.002)^9$; (*b*) $(0.8)^{10}$; (*c*) $(2.002)^8$; (*d*) $(1.997)^{12}$.

CHAPTER XVIII

HYPERBOLIC FUNCTIONS

205. Definitions. Certain combinations of e^x and e^{-x} represent functions analogous to the trigonometric functions. Their geometric representation is related to the equilateral hyperbola in a manner similar to that in which the trigonometric functions are related to the circle. Hence the name ***hyperbolic functions.*** They are defined as follows.

$$
(1) \qquad \begin{cases}
\sinh x = \dfrac{e^x - e^{-x}}{2}, \\[2ex]
\cosh x = \dfrac{e^x + e^{-x}}{2}, \\[2ex]
\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}, \\[2ex]
\operatorname{ctnh} x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}, \\[2ex]
\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}, \\[2ex]
\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}},
\end{cases}
$$

and are read *hyperbolic sine of x*, etc.

206. Functional Relations. From the definitions we have at once the relations

$$
(2) \qquad \begin{cases}
\sinh(-x) = -\sinh x, \\
\cosh(-x) = \cosh x, \\
\tanh(-x) = -\tanh x, \\
\operatorname{ctnh}(-x) = -\operatorname{ctnh} x, \\
\operatorname{sech}(-x) = \operatorname{sech} x, \\
\operatorname{csch}(-x) = -\operatorname{csch} x.
\end{cases}
$$

Squaring and combining so as to eliminate e^x and e^{-x} we obtain the relations

$$(3)\qquad \begin{cases} \cosh^2 x - \sinh^2 x = 1, \\ \tanh^2 x + \operatorname{sech}^2 x = 1, \\ \operatorname{ctnh}^2 x - \operatorname{csch}^2 x = 1. \end{cases}$$

Then from the definitions of $\sinh x$ and $\cosh x$ we have by addition and subtraction

$$(4)\qquad \begin{cases} \cosh x + \sinh x = e^x, \\ \cosh x - \sinh x = e^{-x}. \end{cases}$$

From relations (4) we derive the functions of the sum of two variables as follows:

By definition,

$$\begin{aligned} \sinh (x + y) &= \frac{e^{x+y} - e^{-x-y}}{2} \\ &= \frac{e^x e^y - e^{-x} e^{-y}}{2} \\ &= \tfrac{1}{2}[(\cosh x + \sinh x)(\cosh y + \sinh y) \\ &\quad - (\cosh x - \sinh x)(\cosh y - \sinh y)]. \end{aligned}$$

Whence, expanding and collecting, we have

$$\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y.$$

Replacing y by $-y$ and using relations (2), we get

$$\sinh (x - y) = \sinh x \cosh y - \cosh x \sinh y.$$

This and similar operations give

$$(5)\qquad \begin{cases} \sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y, \\ \cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y, \\ \tanh (x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}, \\ \operatorname{ctnh} (x \pm y) = \dfrac{1 \pm \operatorname{ctnh} x \operatorname{ctnh} y}{\operatorname{ctnh} x \pm \operatorname{ctnh} y}. \end{cases}$$

Letting $y = x$ and using the positive signs in (5), the double variable formulas appear as

$$
(6)\qquad \begin{cases}
\sinh 2x = 2\sinh x\cosh x, \\
\cosh 2x = \cosh^2 x + \sinh^2 x \\
\qquad\quad = 2\cosh^2 x - 1 \\
\qquad\quad = 2\sinh^2 x + 1, \\
\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}, \\
\operatorname{ctnh} 2x = \dfrac{1 + \operatorname{ctnh}^2 x}{2\operatorname{ctnh} x}.
\end{cases}
$$

Then, from the third formula for $\cosh 2x$, we find

$$\sinh x = \pm\sqrt{\frac{\cosh 2x - 1}{2}},$$

or, replacing x by $x/2$ in this and analogous forms, we obtain

$$
(7)\qquad \begin{cases}
\sinh \dfrac{x}{2} = \pm\sqrt{\dfrac{\cosh x - 1}{2}}, \\
\cosh \dfrac{x}{2} = \sqrt{\dfrac{\cosh x + 1}{2}}, \text{ since } \cosh x > 0. \\
\tanh \dfrac{x}{2} = \pm\sqrt{\dfrac{\cosh x - 1}{\cosh x + 1}}, \\
\operatorname{ctnh} \dfrac{x}{2} = \pm\sqrt{\dfrac{\cosh x + 1}{\cosh x - 1}}.
\end{cases}
$$

If we set $P = x + y$ and $Q = x - y$ and use $\sinh(x \pm y)$ and $\cosh(x \pm y)$ as in trigonometry, we have

$$
(8)\qquad \begin{cases}
\sinh P + \sinh Q = 2\sinh\left(\dfrac{P+Q}{2}\right)\cosh\left(\dfrac{P-Q}{2}\right), \\
\cosh P + \cosh Q = 2\cosh\left(\dfrac{P+Q}{2}\right)\cosh\left(\dfrac{P-Q}{2}\right), \\
\sinh P - \sinh Q = 2\cosh\left(\dfrac{P+Q}{2}\right)\sinh\left(\dfrac{P-Q}{2}\right), \\
\cosh P - \cosh Q = 2\sinh\left(\dfrac{P+Q}{2}\right)\sinh\left(\dfrac{P-Q}{2}\right).
\end{cases}
$$

207. Derivatives of the Hyperbolic Functions. Since $\sinh x = (e^x - e^{-x})/2$ we have

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x,$$

and if $u = f(x)$ we have

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}.$$

Thus, and similarly, we get

$$(9)\qquad \begin{cases} \dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}, \\[2ex] \dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}, \\[2ex] \dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}, \\[2ex] \dfrac{d}{dx}(\operatorname{ctnh} u) = -\operatorname{csch}^2 u \dfrac{du}{dx}, \\[2ex] \dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}, \\[2ex] \dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \operatorname{ctnh} u \dfrac{du}{dx}. \end{cases}$$

208. The Inverse Hyperbolic Functions. The inverse of the hyperbolic function is called an ***antihyperbolic function.*** Thus, if

$$x = \sinh y$$

then

$$y = \sinh^{-1} x, \text{ etc.}$$

Since $x = \sinh y = (e^y - e^{-y})/2$, we have

$$2x = e^y - e^{-y},$$

or

$$e^{2y} - 2xe^y - 1 = 0.$$

Therefore

$$e^y = x \pm \sqrt{1 + x^2}.$$

But $e^y > 0$ for all finite values of y, and so the lower sign is impossible; hence

$$y = \sinh^{-1} x = \log (x + \sqrt{1 + x^2}).$$

Thus, and similarly, we have

$$(10) \quad \begin{cases} \sinh^{-1} x = \log (x + \sqrt{1 + x^2}), \\ \cosh^{-1} x = \log (x \pm \sqrt{x^2 - 1}), \quad x \geqq 1, \\ \tanh^{-1} x = \dfrac{1}{2} \log \left(\dfrac{1 + x}{1 - x}\right), \quad -1 < x < 1, \\ \operatorname{ctnh}^{-1} x = \dfrac{1}{2} \log \left(\dfrac{x + 1}{x - 1}\right), \quad |x| > 1, \\ \operatorname{sech}^{-1} x = \log \left(\dfrac{1 \pm \sqrt{1 - x^2}}{x}\right), \quad 0 < x \leqq 1, \\ \operatorname{csch}^{-1} x = \log \left(\dfrac{1 + \sqrt{1 + x^2}}{x}\right), \quad x > 0, \\ \qquad\qquad = \log \left(\dfrac{1 - \sqrt{1 + x^2}}{x}\right), \quad x < 0. \end{cases}$$

By differentiation of the values for the inverse functions we get, since $y = \sinh^{-1} u$,

$$u = \sinh y$$

and then

$$\frac{du}{dx} = \cosh y \frac{dy}{dx}.$$

But $\cosh^2 y - \sinh^2 y = 1$, or $\cosh y = \sqrt{1 + u^2}$. From these relations we have

$$\frac{dy}{dx} = \frac{1}{\cosh y} \frac{du}{dx} = \frac{\dfrac{du}{dx}}{\sqrt{u^2 + 1}}.$$

Similarly we obtain the following formulas:

$$
(11)\qquad
\begin{cases}
\dfrac{d}{dx}(\sinh^{-1} u) = \dfrac{\dfrac{du}{dx}}{\sqrt{u^2+1}}, \\[2ex]
\dfrac{d}{dx}(\cosh^{-1} u) = \pm\dfrac{\dfrac{du}{dx}}{\sqrt{u^2-1}}, \\[2ex]
\dfrac{d}{dx}(\tanh^{-1} u) = \dfrac{\dfrac{du}{dx}}{1-u^2}, \\[2ex]
\dfrac{d}{dx}(\operatorname{ctnh}^{-1} u) = \dfrac{\dfrac{du}{dx}}{1-u^2}, \\[2ex]
\dfrac{d}{dx}(\operatorname{sech}^{-1} u) = \mp\dfrac{\dfrac{du}{dx}}{u\sqrt{1-u^2}}, \\[2ex]
\dfrac{d}{dx}(\operatorname{csch}^{-1} u) = -\dfrac{\dfrac{du}{dx}}{\sqrt{u^2+u^4}}.
\end{cases}
$$

209. Integration by Hyperbolic Substitutions. The derivative formulas (11) suggest some integration formulas which may be very easily derived by hyperbolic substitutions. For example, consider

$$\int \frac{dx}{\sqrt{x^2+a^2}}.$$

Set $x = a \sinh u$, then $dx = a \cosh u \, du$ and

$$\sqrt{x^2+a^2} = \sqrt{a^2 \sinh^2 u + a^2} = a \cosh u.$$

So the integral becomes

$$\int \frac{a \cosh u \, du}{a \cosh u} = \int du = u + C = \sinh^{-1}\frac{x}{a} + C.$$

This and similar substitutions enable us to obtain readily

$$
(12) \quad \left\{
\begin{aligned}
\int \frac{dx}{\sqrt{x^2 + a^2}} &= \sinh^{-1}\frac{x}{a} + C, \\
\int \frac{dx}{\sqrt{x^2 - a^2}} &= \pm \cosh^{-1}\frac{x}{a} + C, \\
\int \frac{dx}{a^2 - x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \\
\int \frac{dx}{x\sqrt{a^2 - x^2}} &= \pm \frac{1}{a}\operatorname{sech}^{-1}\frac{x}{a} + C \\
&= \pm \frac{1}{a}\cosh^{-1}\frac{a}{x} + C, \\
\int \frac{dx}{x\sqrt{a^2 + x^2}} &= -\frac{1}{a}\operatorname{csch}^{-1}\frac{x}{a} + C \\
&= -\frac{1}{a}\sinh^{-1}\frac{a}{x} + C.
\end{aligned}
\right.
$$

The relations (**10**) allow us to write formulas (**12**) with logarithmic results.

210. Series for Hyperbolic Functions. Since the Maclaurin expansions of e^x and e^{-x} are

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots,$$

we may write

$$\sinh x = \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\right],$$

or

$$(13) \qquad \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

In like manner

(14) $$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

We have previously assumed (§ 204) that

$$\sin(ix) = ix - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} - \cdots$$
$$= i\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right).$$

Therefore

(15) $$\sin(ix) = i \sinh x.$$

Similarly

(16) $$\cos(ix) = \cosh x.$$

Also the relations

(17) $$\cosh(ix) = \cos x,$$

and

(18) $$\sinh(ix) = i \sin x$$

are readily derived.

211. The Relation to the Equilateral Hyperbola. The circle with its center at the origin and radius a may be represented parametrically by

$$\begin{cases} x = a\cos\theta, \\ y = a\sin\theta. \end{cases}$$

The sector OAB has its area K represented by

$$a^2\theta_1/2, \quad \text{where} \quad \theta_1 = \cos^{-1}(x_1/a).$$

So $\cos^{-1}(x_1/a) = 2K/a^2$. For the unit circle the parameter θ becomes $2K$ or ***twice the area of the sector.***

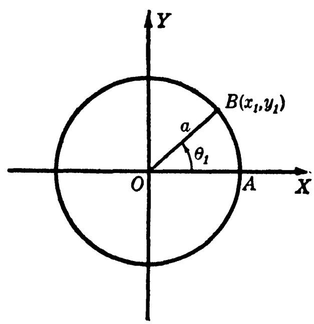


FIG. 223

The equilateral hyperbola with its center at the origin and semi-transverse axis a along the x axis, may be represented parametrically by the equations

$$\begin{cases} x = a\cosh u, \\ y = a\sinh u, \end{cases}$$

since, eliminating u by means of $\cosh^2 u - \sinh^2 u = 1$, we have $x^2 - y^2 = a^2$.

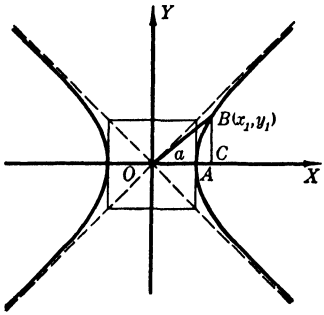


FIG. 224

Now consider the hyperbolic sector AOB. Its area is

$$K = \frac{x_1 y_1}{2} - \int_a^{x_1} \sqrt{x^2 - a^2}\,dx$$

$$= \frac{x_1 y_1}{2} - \frac{1}{2}\Big[\sqrt{x^2 - a^2}$$

$$- a^2 \log (x + \sqrt{x^2 - a^2})\Big]_a^{x_1}$$

$$= \frac{x_1 y_1}{2} - \frac{x_1\sqrt{x_1^2 - a^2}}{2} + \frac{a^2}{2}[\log (x_1 + \sqrt{x_1^2 - a^2}) - \log a]$$

$$= \frac{x_1 y_1}{2} - \frac{x_1 y_1}{2} + \frac{a^2}{2} \log \left(\frac{x_1}{a} + \frac{\sqrt{x_1^2 - a^2}}{a}\right),$$

whence, by the second of relations (10), we have

$$K = \frac{a^2}{2} \cosh^{-1} \frac{x_1}{a}.$$

This may be written

$$\cosh^{-1} \frac{x_1}{a} = u_1 = \frac{2\,K}{a^2},$$

and for $a = 1$ the parameter u becomes $2\,K$, ***just as the parameter does in the circle.***

We note then two things:

(*a*) *The* $\cos^{-1}(x/a)$, *or* θ, *the parameter in the equations of the circle, is represented by twice the area of the circular sector divided by the square of the radius* a.

(*b*) *The* $\cosh^{-1}(x/a)$, *or* u, *the parameter in the equations of the equilateral hyperbola, is represented by twice the area of the hyperbolic sector divided by the square of the semi-transverse axis* a.

212. Relations between Hyperbolic Functions of u and Trigonometric Functions of θ. Such relations are derived at once from Fig. 225. Here BC is tangent to the circle at B, and CP is per-

pendicular to OA. So to each θ there is a point B on the circle which fixes a point P on the hyperbola. The parametric equations of the hyperbola given in the preceding article and Fig. 225 give

$$a \cosh u = OC = OB \sec \theta = a \sec \theta.$$

The definitions of the hyperbolic functions and the identities (**3**) of the first article of this chapter together with

$$\cosh u = \sec \theta$$

give

$$\sinh u = \tan \theta,$$
$$\tanh u = \sin \theta,$$
$$\operatorname{ctnh} u = \csc \theta,$$
$$\operatorname{csch} u = \operatorname{ctn} \theta,$$
$$\operatorname{sech} u = \cos \theta.$$

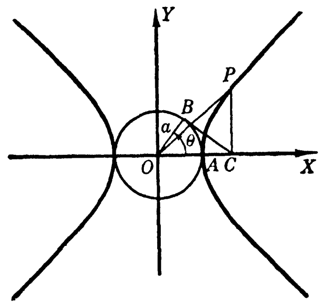


FIG. 225

In these relations the variable θ is called the ***gudermannian*** of u and is denoted by

$$\theta = \operatorname{gd} u.$$

PROBLEMS

1. All formulas not expressly derived in this chapter may be used as exercises.

2. Derive

(a) $\sinh (x \pm iy)$, (b) $\cosh (x \pm iy)$, (c) $\tanh (x \pm iy)$,

in terms involving $\sinh x$, $\cosh x$, $\sinh y$, $\cosh y$, $\sin x$, $\cos x$, $\sin y$, $\cos y$.

3. Show that
(a) $(\cosh x + \sinh x)^5 = \cosh 5x + \sinh 5x$.
(b) $(\cosh x + \sinh x)^k = \cosh kx + \sinh kx$.

4. Derive $\sinh (x - iy)$ and evaluate for $x = 3$, $y = 3$.

5. Derive $\cosh (x - iy)$ and evaluate for $x = 4.2$, $y = 3$.
Ans. $-32.9 - 4.70\,i$.

6. Derive $\cosh (x - iy)$ and evaluate for $x = 4$, $y = 3$.

7. Derive $\cosh(x - iy)$ and evaluate for $x = 3$, $y = -2$.
Ans. $-4.19 + 9.11\,i$.

8. Derive $\cosh(x - iy)$ and evaluate for $x = 2$, $y = 3$.

9. Problem 4 for $x = 1$, $y = 3$. *Ans.* $-1.16 - 0.22\,i$.

10. Problem 4 for $x = 1.6$, $y = 0.8$.

11. Problem 4 for $x = 4$, $y = 0.5$. *Ans.* $23.96 - 13.08\,i$.

12. Problem 4 for $x = 3$, $y = -2$.

Integrate each of the following integrals by means of hyperbolic substitutions. (Nos. 13–23.)

13. $\displaystyle\int \frac{dx}{\sqrt{4 + x^2}}$. *Ans.* $\sinh^{-1}(x/2) + C$.

14. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 3}}$.

15. $\displaystyle\int \frac{dx}{x^2 - 3}$. *Ans.* $\displaystyle -\frac{1}{\sqrt{3}}\tanh^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$.

16. $\displaystyle\int \frac{dx}{x\sqrt{4 - x^2}}$.

17. $\displaystyle\int \frac{dx}{x\sqrt{4 + x^2}}$. *Ans.* $\displaystyle -\frac{1}{2}\sinh^{-1}\left(\frac{2}{x}\right) + C$.

18. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 4x + 1}}$.

19. $\displaystyle\int \frac{dx}{(x + 4)\sqrt{9 - 8x - x^2}}$. *Ans.* $\displaystyle \pm\frac{1}{5}\cosh^{-1}\frac{5}{x + 4} + C$.

20. $\displaystyle\int \frac{dx}{2 - x^2 - 5x}$.

21. $\displaystyle\int \frac{dx}{(x - 1)\sqrt{x^2 - 2x + 26}}$. *Ans.* $\displaystyle -\frac{1}{5}\sinh^{-1}\frac{5}{x - 1} + C$.

22. $\displaystyle\int \frac{dx}{x\sqrt{2 - x^2 - 4x}}$.

23. $\displaystyle\int \frac{dx}{x\sqrt{14 + x^2 - 6x}}$. *Ans.* $\displaystyle -\frac{1}{\sqrt{14}}\sinh^{-1}\frac{14 - 3x}{x\sqrt{5}} + C$.

CHAPTER XIX

EXACT DIFFERENTIALS AND LINE INTEGRALS

213. Exact Differentials. A function of two variables, say $f(x, y)$, has as its differential the known expression

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy, \tag{1}$$

where $f(x, y)$ and its partial derivatives are supposed to be continuous in the interval under consideration.

However, every expression

$$f_1(x, y)dx + f_2(x, y)dy \tag{2}$$

of the same type as (1) need not be the differential of some function $F(x, y)$. Writing (2) in the form

$$M\,dx + N\,dy, \tag{3}$$

it is the differential of some $F(x, y)$ if $M = \partial F/\partial x$ and $N = \partial F/\partial y$, since

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy.$$

But, under our assumptions of continuity,

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right);$$

hence we find that M and N must satisfy the relation

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \tag{4}$$

if (3) represents the differential of $F(x, y)$.

It can be shown that not only is (4) necessary for (3) to represent the differential of some function, but it is also sufficient. That is, *if M and N satisfy* (4) *there is always an $F(x, y)$ for which* (3) *is the differential.**

* Goursat-Hedrick, *Mathematical Analysis*, Vol. I, §§ 151–152.

If we have an exact differential, it is often desirable to find the original function. An example will illustrate a method of finding the function.

EXAMPLE

Is $(2x + 5y - 7)dx + (5x - 8y + 3)dy$ an exact differential? If so, find the original function.

SOLUTION. Here $M = 2x + 5y - 7$ and $N = 5x - 8y + 3$. Since $\partial M/\partial y = 5 = \partial N/\partial x$, the expression is exact. Therefore

$$M = \frac{\partial F}{\partial x} = 2x + 5y - 7,$$

which we obtain by differentiation with respect to x, assuming y constant. Then, integrating with respect to x, regarding y as constant, we get

$$F(x, y) = x^2 + 5xy - 7x + f(y),$$

where the constant of integration is represented by $f(y)$, since y is assumed constant. Now differentiate this expression for $F(x, y)$ with respect to y. Then

$$\frac{\partial F}{\partial y} = 5x + \frac{d}{dy}f(y).$$

Since $N = \partial F/\partial y$, we have

$$5x + \frac{d}{dy}f(y) = 5x - 8y + 3,$$

or

$$\frac{d}{dy}f(y) = -8y + 3.$$

Therefore

$$f(y) = -4y^2 + 3y + C,$$

and we have

$$F(x, y) = x^2 + 5xy - 7x - 4y^2 + 3y + C.$$

PROBLEMS

Integrate the differentials below if they are exact.

1. $x\,dy + (y - 7)dx.$ *Ans.* $xy - 7x + C.$

2. $(3y - 8)dx + (3x + 7)dy.$

3. $(x + y)dx + (x - y)dy.$ *Ans.* $(x^2 - y^2)/2 + xy + C.$

4. $(3x - 2y)dx + (2x + 3y)dy$.

5. $(xy \cos xy + \sin xy)dx + x^2 \cos xy\, dy$. *Ans.* $x \sin xy + C$.

6. $(e^x \cos y - 1)dx - e^x \sin y\, dy$.

7. $(e^y/x)dx + e^y \log x\, dy$. *Ans.* $e^y \log x + C$.

8. $(x - 1/y)dy + (y - 1/x)dx$.

9. $(xy + x^2)dx + (x^2/2)dy$. *Ans.* $x^3/3 + x^2y/2 + C$.

10. $(\sin^2 x - y \cos x)dx + \sin x\, dy$.

11. $(e^x \sin y - y)dx + (e^x \cos y - x - 2)dy$.
Ans. $e^x \sin y - xy - 2y + C$.

12. $(x/\sqrt{x^2 + y^2})dx + (y/\sqrt{x^2 + y^2})dy$.

13. $e^x \cos x\, dy + ye^x \sin x\, dx$. *Ans.* Not exact.

14. $xe^{-x/y}\, dy - (ye^{-x/y} + x^2)dx$.

15. $dx/\sqrt{x^2 + y^2} + [y/(x\sqrt{x^2 + y^2} + x^2 + y^2)]dy$.
Ans. $\log (x + \sqrt{x^2 + y^2}) + C$.

16. $(xy + x \sin xy)dy + (y^2/2 + y \sin xy + x)dx$.

17. $(1/x + y/x^2e^{y/x})dx - dy/xe^{y/x}$. *Ans.* $\log x + e^{-y/x} + C$.

18. $dx/\sqrt{x^2 - y^2} + [(x - \sqrt{x^2 - y^2})/y]dy$.

19. $xe^{xy} \sin y\, dx + (e^{xy} \cos y + y)dy$. *Ans.* Not exact.

20. $[(2xy + 1)/y]dx + [(y - x)/y^2]dy$.

214. Line Integrals. The expression

$$\int_{(a,\, b)}^{(c,\, d)} [f_1(x, y)dx + f_2(x, y)dy]$$

is called a ***line integral*** and has a meaning when some relation between x and y is known. This relation, say $y = F(x)$, must represent a curve through the points (a, b) and (c, d) used as the limits of the integral. In general, the value of a line integral changes when the line or curve $y = F(x)$ of integration is changed, but if $f_1(x, y)dx + f_2(x, y)dy$ is exact, the integral has the same

value for all curves joining (a, b) and (c, d). This is evident because in that case

$$\int_{(a,b)}^{(c,d)} [f_1(x, y)dx + f_2(x, y)dy] = \int_{z_1}^{z_2} dz = z_2 - z_1,$$

where z is the integral of the exact differential expression $f_1(x, y)dx + f_2(x, y)dy$ and z_2 and z_1 are determined by the points (a, b) and (c, d).

EXAMPLES

1. Assume that at each point of the xy plane there acts a force which varies as we go from point to point. Let us find the work done in moving a body along the curve C from A to B.

SOLUTION. Let PT be the tangent to the curve C at any point P, and let PR be the direction of the force F at P. Then the work done in moving the body a distance Δs_i is given approximately by $F_i \cos \theta_i \cdot \Delta s_i$,* so that the work in going from A to B along C is

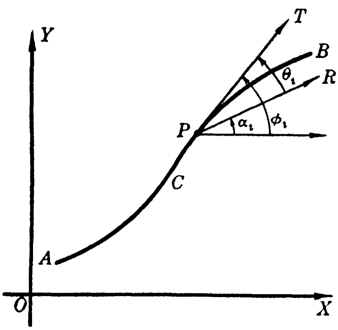


FIG. 226

$$W = \lim_{\substack{n \to \infty \\ \Delta s_i \to 0}} \sum_{i=1}^{n} F_i \cos \theta_i \cdot \Delta s_i$$

$$= \int_C F \cos \theta \, ds,$$

where $\int_C$ means the integral along the curve C, that is, the variables in the integral are related by means of the equation of C.

Therefore we have

$$W = \int_C F(\cos \phi \cos \alpha + \sin \phi \sin \alpha) ds,$$

since $\theta = \phi - \alpha$. But the x-component of the force F is $F \cos \alpha$, which we shall represent by F_x; and the y-component is $F_y = F \sin \alpha$. Also $\cos \phi \, ds = dx$ and $\sin \phi \, ds = dy$. Therefore

$$W = \int_C (F_x dx + F_y dy),$$

or the work is a line integral along the curve C.

* Work is defined as *the product of the force exerted and the distance the force acts in a given direction.*

The point A may coincide with B, thus making C a closed curve; in this case, if $F_x dx + F_y dy$ is an exact differential, it is easy to see that

$$\int_C (F_x dx + F_y dy) = 0.$$

2. Derive a formula employing a line integral for the area bounded by a closed curve.

SOLUTION. Consider the curve below which has not more than two values of y for each value of x and not more than two values of x for each value of y. Now let $NN_1 = y_1$, $NN_2 = y_2$, $SS_1 = x_1$, and $SS_2 = x_2$. Then the area A of the portion of the plane enclosed by the curve is

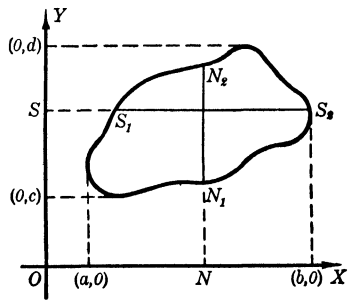


FIG. 227

$$\begin{aligned} A &= \int_a^b (y_2 - y_1)dx \\ &= \int_a^b y_2 dx - \int_a^b y_1 dx \\ &= -\int_b^a y_2 dx - \int_a^b y_1 dx \\ &= -\int_C y\, dx, \end{aligned}$$

this last integral being taken around the curve so that the area lies to the left. Similarly, we have

$$\begin{aligned} A &= \int_c^d (x_2 - x_1)dy \\ &= \int_c^d x_2 dy + \int_d^c x_1 dy \\ &= \int_C x\, dy, \end{aligned}$$

where again the integral is taken so that the area lies to the left.

Adding the two values for A, we have

$$A = \frac{1}{2}\int_C (x\, dy - y\, dx),$$

a line integral giving the area of the closed curve.

3. Apply the formula of Example 2 to find the area of the circle $x = a \cos \theta$, $y = a \sin \theta$.

SOLUTION. Here $dx = -a \sin \theta\, d\theta$ and $dy = a \cos \theta\, d\theta$. Therefore the formula above becomes

$$A = \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} a^2(\cos^2 \theta + \sin^2 \theta)d\theta = \frac{a^2}{2}\int_0^{2\pi} d\theta = \pi a^2.$$

PROBLEMS

Find the values of each of the following line integrals. (Nos. 1–3.)

1. $\int_{(0,0)}^{(4,4)} [2\,xy\,dx + (x^2 - y^2)dy]$ along (*a*) a straight line; (*b*) a parabola with its axis on the x axis; (*c*) a parabola with its axis on the y axis. Show that the integral is independent of the path. *Ans.* 128/3; $M_y = N_x$.

2. $\int_{(0,0)}^{(4,4)} (y\,dx - x\,dy)$ along the paths used in Problem 1.

3. $\int_{(0,0)}^{(4,2)} [x\,dx/\sqrt{x^2+y^2} + y\,dy/\sqrt{x^2+y^2}]$ along (*a*) a straight line; (*b*) $x = t^2$, $y = t$; (*c*) a broken line consisting of a part of the x axis and a perpendicular to it. *Ans.* $2\sqrt{5}$.

4. Find the area of an ellipse by means of a line integral.

5. By means of a line integral, find the area enclosed by the hypocycloid $x = a\cos^3\theta$, $y = a\sin^3\theta$. *Ans.* $3\,\pi a^2/8$ sq. units.

6. Evaluate $\int (1/2)(x\,dy - y\,dx)$ along (*a*) the parabola $y^2 = 4\,x$ from (1, 2) to (0, 0); (*b*) the straight line from (0, 0) to (1, 2). *The difference of these two values gives the area enclosed by the parabola and the line.*

7. Find the area of the ellipse $x = 2\cos\theta$, $y = 4\sin\theta$.
Ans. $8\,\pi$ sq. units.

8. Find the area enclosed by the curve $x = a(1 - \cos\theta)$, $y = a\sin\theta$.

9. Evaluate $\int (xy\,dx - y^2dy)$ from (1, 1) to (4, − 8) along the curve $x = t^2$, $y = t^3$. *Ans.* $134\frac{1}{4}$.

ADDITIONAL PROBLEMS

Integrate each of the following functions if it is exact. (Nos. 1–9.)

1. $(x + y + a)dx + (x - y + b)dy$.
Ans. $x^2/2 - y^2/2 + xy + ax + by + C$.

2. $(\cos y + 2\,x)dx - x\sin y\,dy$.

3. $(x^2 + y^2)dy + 2\,xy\,dx$. *Ans.* $x^2y + y^3/3 + C$.

4. $2\,x\,dy - y^3dy + 2\,y\,dx$.

5. $x(x + 2\,y)dx + (x^2 - y^2)dy$. *Ans.* $x^3/3 - y^3/3 + x^2y + C$.

6. $(e^x - a)dy - e^xy\,dx$.

7. $(2y^2 - 3x)dx - 4xy\,dy$. *Ans.* Not exact.

8. $(2x + y/x^2)dx - (1/x + 2\cos 2y)dy$.

9. $\dfrac{x\,dx}{\sqrt{x^2 - y^2}} + \dfrac{y\,dy}{\sqrt{x^2 - y^2}}$. *Ans.* Not exact.

10. Find the value of $\int_{(0,0)}^{(2,4)} (x^2dx + y\,dy)$ along the curve $x = t,\ y = t^2$.

11. Find the value of $\int_{(2,0)}^{(2\sqrt{2},2)} (y\,dx - x\,dy)/\sqrt{x^2 - y^2}$ along the curve $x^2 - y^2 = 4$. *Ans.* $-2\log(1 + \sqrt{2})$.

12. Determine the value of

$$\int_{(0,0)}^{(1,1)} [(2x + y\,e^{xy})dx + (\cos y + x\,e^{xy})dy]$$

along a straight line joining the two limits. Is this value independent of the path?

13. Evaluate $\int_{(0,0)}^{(1,2)} [(xy + x^2)dx + x^2dy]$, (*a*) along a line through the origin, (*b*) along a parabola with its vertex at the origin and its axis along the y axis. Is the integral independent of the path? *Ans.* 5/3; 11/6. No.

14. Find the value of $\int_{(0,0)}^{(2,1)} [(y - x)dx + y\,dy]$, (*a*) along a straight line, (*b*) along a broken line consisting of a part of the y axis and a line perpendicular to it, (*c*) along a broken line consisting of a part of the x axis and a line perpendicular to it.

CHAPTER XX

SOME DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

215. Definitions. In many cases it is easier to find relations between the rates of change of functions than between the actual functions. Such relations are equations which involve differentials or derivatives, and are called ***differential equations.*** A differential equation which has a single independent variable is called an ***ordinary differential equation.*** If several independent variables occur, so that partial derivatives are present, the equation is known as a ***partial differential equation.***

The ***order*** of a differential equation is the same as that of the derivative of highest order in it.

The ***degree*** is the degree of the derivative of highest order after the equation has been made rational and integral in the derivatives. Thus

$$\frac{d^2y}{dx^2} = \left(y + \frac{dy}{dx}\right)^{1/2}$$

is an ordinary differential equation of the *second order* and of the *second degree.* The equation

$$\frac{d^3y}{dx^3} + 2\left(\frac{dy}{dx}\right)^2 = 6$$

is of the *third order* and *first degree.*

An illustration of a partial differential equation of the *second order* is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

216. Derivation of a Differential Equation from Its Primitive. The equation $y^2 = 2\,kx$ represents a *one-parameter family of parabolas,* since k may be considered to have any value whatever. If we differentiate this equation with respect to x, we have

$2\,y\,dy = 2\,k\,dx$. Eliminating k between the two relations, we find the first-order differential equation

$$2\,x\,dy = y\,dx. \tag{1}$$

This equation is called ***the*** differential equation which arises from eliminating the arbitrary constant k between $y^2 = 2\,kx$ and $2\,y\,dy = 2\,k\,dx$. It must represent some characteristic common to all of the parabolas of the family, since it is independent of k.

Similarly, suppose we have

$$y = c_1 \sin x + c_2 \cos x \tag{2}$$

and wish to eliminate the constants c_1 and c_2. To do this we need two additional equations which may be derived by repeated differentiation. Thus

$$\frac{dy}{dx} = c_1 \cos x - c_2 \sin x, \tag{3}$$

and

$$\frac{d^2y}{dx^2} = -\,c_1 \sin x - c_2 \cos x. \tag{4}$$

Eliminating c_1 and c_2 between equations (2), (3), and (4), we get

$$\frac{d^2y}{dx^2} + y = 0.$$

The equation (2) with two constants gives rise to a differential equation of the second order.

If we have an equation in x and y involving three arbitrary constants we need three additional equations to eliminate the constants. These are obtained by repeated differentiation and will introduce derivatives of the third order.

The original equation connecting x and y is called the ***primitive*** of the derived differential equation. Thus

$$y = c_1 \sin x + c_2 \cos x$$

is the primitive of the equation

$$\frac{d^2y}{dx^2} + y = 0.$$

By extending this reasoning, it appears that *a primitive involving n arbitrary constants gives rise to a differential equation of the n-th order.* Also, *that a differential equation of the n-th order cannot have more than n arbitrary constants in its solution.* For if there were more, say $n + 1$, elimination of them would seem to lead to a differential equation of order $n + 1$ instead of n. Similarly, it cannot have less than n constants. *Hence an equation of order n has a solution or primitive with n arbitrary constants.*

PROBLEMS

Form the differential equation of lowest order which arises from elimination of constants in each of the following primitives.

1. $y = cx - \sqrt{1 + c^2}$. *Ans.* $y = x\dfrac{dy}{dx} - \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$.

2. $\log x + \log(y + 1) = c$.

3. $y = cx^2$. *Ans.* $x\,dy - 2\,y\,dx = 0$.

4. $y^2 = 2\,cx + c^2$.

5. $\log x + e^{y/x} = c$. *Ans.* $x\,dx + (x\,dy - y\,dx)e^{y/x} = 0$.

6. $x = c_1y^2 + c_2y$.

7. $y = c_1 \sin x + c_2 \cos x + x^2$. *Ans.* $\dfrac{d^2y}{dx^2} + y = x^2 + 2$.

8. $y = c_1 + c_2e^{3x} + x^2$.

9. $ye^x = c_1x^2 + c_2x + c_3$. *Ans.* $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = 0$.

10. $(x - h)^2 + (y - k)^2 = r^2$.

11. $x - y^2 = c_1x^2 + c_2$. *Ans.* $2\,xy\dfrac{d^2y}{dx^2} + 2\,x\left(\dfrac{dy}{dx}\right)^2 - 2\,y\dfrac{dy}{dx} + 1 = 0$.

12. $y = c_1e^x + c_2e^{3x} + c_3$.

217. Solutions of a Differential Equation. A ***solution*** or ***integral*** of a differential equation is any relation between the variables which satisfies the equation. Thus $y = \cos x + c$ is a solution of $dy + \sin x\,dx = 0$; and, as c may have any value whatever, the differential equation has an *infinite number of solutions.* Each solution may be represented by a curve in the xy plane called an ***integral curve.***

The second order equation

$$\frac{d^2y}{dx^2} + y = 0$$

has the solution $y = c_1 \cos x + c_2 \sin x$, where c_1 and c_2 are *arbitrary constants.* This solution may evidently represent a double infinity of integral curves.

A solution of a differential equation which involves the maximum number of arbitrary constants is called the ***general*** or ***complete solution.*** Solutions obtained by giving particular values to the arbitrary constants of the general solution are called ***particular solutions.*** Thus $y = c_1 \sin x + c_2 \cos x$ is the general solution of $d^2y/dx^2 + y = 0$, while each of the equations $y = 3 \sin x$ and $y = 2 \sin x - \cos x$ are particular solutions.

Methods of solving the more general types of the first order and first degree follow.

218. Variables Separable. The first-order equation of first degree can always be written in the form

$$M\,dx + N\,dy = 0. \tag{1}$$

If it is of the type where *variables are separable,* each of the quantities M and N can be resolved into two factors such that x does not occur in one factor and y does not occur in the other.

That is, we may write

$$M = f_1(x) \cdot f_2(y), \qquad N = f_3(x) \cdot f_4(y), \tag{2}$$

where any of the functions f_i may be constants. Then (1) becomes

$$f_1(x) \cdot f_2(y)dx + f_3(x) \cdot f_4(y)dy = 0.$$

Dividing through by $f_2(y) \cdot f_3(x)$, we get an equation of the form

$$f(x)dx + g(y)dy = 0. \tag{3}$$

The solution of (3) is given at once by

$$\int f(x)dx + \int g(y)dy = c. \tag{4}$$

EXAMPLE

Solve the equation $x\sqrt{1 + y^2}\,dx + y\sqrt{1 + x^2}\,dy = 0$.

SOLUTION. Here $M = x\sqrt{1 + y^2}$, $N = y\sqrt{1 + x^2}$ where each has two factors of the types necessary for separation of the variables. Whence dividing by $\sqrt{1 + y^2} \cdot \sqrt{1 + x^2}$ we have

$$\frac{x\,dx}{\sqrt{1 + x^2}} + \frac{y\,dy}{\sqrt{1 + y^2}} = 0.$$

Integration of each term gives the solution as

$$\sqrt{1 + x^2} + \sqrt{1 + y^2} = c.$$

219. Exact Differential Equations. An ***exact differential*** equation is one which has been formed from its primitive by differentiation without any additional operations of reduction. The test for exactness and the method of integration are given in § 213 on exact differentials. *The solution of an exact differential equation is the integral of the exact differential set equal to a constant.*

EXAMPLE

The equation $x\,dy + y\,dx - 2\,x\,dx = 0$ has the solution $xy - x^2 = c$.

220. Homogeneous Equations. If $f(x, y)$ becomes $x^n \cdot F(v)$ when we set $y = vx$, then $f(x, y)$ is said to be ***homogeneous*** in x and y of the n-th degree. Thus $e^{x/y}$ is homogeneous of degree zero, since $e^{x/y} = e^{x/vx} = e^{1/v} = x^0 \cdot e^{1/v}$. The function $(xy + y^2)^{1/2}$ becomes $x(v + v^2)^{1/2}$ and is accordingly homogeneous of the first degree.

If M and N are homogeneous functions of the same degree the equation $M\,dx + N\,dy = 0$ is ***homogeneous.*** *In that case either $y = vx$ or $x = vy$ will make the variables separable.* This is because the equation becomes, for $y = vx$,

$$x^n \cdot f_1(v)dx + x^n \cdot f_2(v)(v\,dx + x\,dv) = 0,$$

or

$$[f_1(v) + v\,f_2(v)]dx + x\,f_2(v)dv = 0;$$

and if the function $f_1(v) + v\,f_2(v) \neq 0$ the result may be written

$$\frac{dx}{x} + \frac{f_2(v)dv}{f_1(v) + v\,f_2(v)} = 0,$$

where the variables have been separated. If the bracket is zero the equation reduces to $dv = 0$; hence $v = c$ or $y = cx$ is a solution of the equation.

Sometimes the substitution $y = vx$ gives a function of v which is not readily integrated, when $x = vy$ will give a more desirable function.

EXAMPLE

Solve the equation $(x^2 + xy)dy - y^2dx = 0.$

Solution. Let $x = vy$; then $dx = v\,dy + y\,dv$. Therefore

$$(v^2y^2 + vy^2)dy - y^2(v\,dy + y\,dv) = 0,$$

or

$$\frac{dy}{y} = \frac{dv}{v^2}.$$

Integrating, we have

$$\log y = -\frac{1}{v} + \log c,$$

or

$$\log \frac{y}{c} = -\frac{1}{v}.$$

Whence $$y = ce^{-y/x}, \text{ since } v = x/y.$$

If the integration introduces a logarithm it is desirable to call the constant of integration $\log c$ so that the two logarithms may be combined as shown in the solution.

PROBLEMS

Integrate each of the following equations. (Nos. 1–19.)

1. $(x - xy)dy + (y + xy)dx = 0.$ *Ans.* $\log xy + x - y = c.$

2. $x\sqrt{1 - y^2}\,dx - y\sqrt{1 - x^2}\,dy = 0.$

3. $(6x - 2y + 1)dx - (2x - 2y + 3)dy = 0.$
Ans. $3x^2 + y^2 - 2xy + x - 3y = c.$

4. $x^2\sqrt{1 + y}\,dx + y^2\sqrt{1 - x}\,dy = 0.$

5. $(y - x^2 - 1)dx + x\,dy = 0.$ *Ans.* $3xy - x^3 - 3x = c.$

6. $6xy\,dx + 3x^2\,dy - 4x^3y\,dx = x^4\,dy - 15y^2\,dy.$

7. $(2y\cos 2x + 2\sin 2x)dx + \sin 2x\,dy = 0.$
Ans. $y \sin 2x - \cos 2x = c.$

8. $(y^2 - xy)dx + (x^2 + xy)dy = 0.$

9. $(y^2e^{xy} + \cos x)dx + (2y + e^{xy} + xy \cdot e^{xy})dy = 0.$
Ans. $ye^{xy} + \sin x + y^2 = c.$

10. $x^2\,dy - xy\,dx = (y^2 - xy)dy.$

11. $2xy\,dy = (1 - y^2)dx.$ *Ans.* $\log x(1 - y^2) = c.$

12. $(6x - 2y + 5)dx + (2y - 2x + 3)dy = 0.$

13. $2y^2\,dx + (4xy - 3y^2)dy = 0.$ *Ans.* $2xy^2 - y^3 = c.$

14. $(3x - 4y)dx - (4x + 7y)dy = 0.$

15. $e^y\,dx - \sin x\,dx + xe^y\,dy = 0.$ *Ans.* $xe^y + \cos x = c.$

16. $[e^x(1 + x) + y\cos xy]dx + (4y + x\cos xy)dy = 0.$

17. $x^2\,dy + y^2\,dx + 2xy(dx + dy) - 3x^2\,dx = 0.$
Ans. $x^2y + xy^2 - x^3 = c.$

18. $(ye^x + 2x)dx - (2 - e^x)dy = 0.$

19. $(2x - 3y - y\sin x)dx + (\cos x - 3x)dy = 0.$
Ans. $x^2 - 3xy + y\cos x = c.$

20. The rate of change of the population of a city is proportional to the population at any time. Find the equation expressing its growth. If the population doubles in 60 years, how long will it take to treble?

21. The tangent to a member of a system of curves has its x intercept equal to three times the abscissa of the contact. What is the system?
Ans. $xy^2 = c.$

Orthogonal Trajectories. Let $F(x, y, dy/dx) = 0$ be the differential equation obtained from the one-parameter family of curves $f(x, y, c) = 0$. Then the differential equation $F(x, y, -dx/dy) = 0$ has for its solution a one-parameter family known as the *orthogonal trajectories* of the given system, since each curve of one system meets every curve of the other system at right angles. Find the orthogonal trajectories of each of the following families. (Nos. 22–23.)

22. The parabolas $y^2 = kx.$

23. The system of circles $x^2 + y^2 - 2kx = 0.$ *Ans.* $x^2 + y^2 - cy = 0.$

24. Prove that the conics $\dfrac{x^2}{a^2 + k} + \dfrac{y^2}{b^2 + k} = 1$, where a and b are fixed from a self-orthogonal system.

25. By solving a first-order differential equation, show that the diameters of the system of circles $x^2 + y^2 = a^2$ are the orthogonal trajectories.

26. Solve $dy/dx + xy = x$ completely for the curve through (1, 1).

27. The length of the tangent drawn from any point $P(x, y)$ of a tractrix is a units. Show that the differential equation of the curve is $\sqrt{a^2 - y^2}\,dy = y\,dx$. If $(0, a)$ is a point of the curve, prove that its equation is

$$x = a \log \frac{a - \sqrt{a^2 - y^2}}{y} + \sqrt{a^2 - y^2}.$$

221. Linear Equations of the First Order. A ***linear differential equation*** is one that is of the first degree in the dependent variable y and its derivatives. Thus

$$P_1 \frac{d^2y}{dx^2} + P_2 \frac{dy}{dx} + P_3 y = f(x)$$

is a linear equation in y of the second order if P_1, P_2, and P_3 are functions of x only or are constants. We are at present interested only in linear equations of the first order, the general type for which is

$$(1) \qquad \frac{dy}{dx} + Py = Q,$$

where P and Q are functions of x or are constants.

The substitution $y = uv$ permits the solution of this equation; the method being indicated in the examples given below.

In the equation $M\,dx + N\,dy = 0$, of course, either variable may be considered the dependent variable. Therefore, an equation may sometimes be put in the form

$$(2) \qquad \frac{dx}{dy} + P'x = Q',$$

where P' and Q' are functions of y or are constants. Then we use the substitution $x = uv$ and the method is the same in both cases.

EXAMPLES

1. Solve $(x + 1)\,dy - 2\,y\,dx = (x + 1)^4\,dx$.

SOLUTION. Since this equation is of the first degree in y and dy, we write it in the form (1) of this article. Whence we get

$$(1) \qquad \frac{dy}{dx} - \left(\frac{2}{x + 1}\right) y = (x + 1)^3.$$

Now set

$$(2) \qquad y = uv;$$

then

$$(3) \qquad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Substituting (2) and (3) in (1) and collecting the coefficients of v and dv/dx, we have

$$(4) \qquad u\frac{dv}{dx} + \left(\frac{du}{dx} - \frac{2u}{x+1}\right)v = (x+1)^3.$$

Now the relation (2) introduces two unknowns and hence two conditions are needed to determine them. One condition is necessarily that $y = uv$ shall satisfy equation (1); the other we may impose at will. We shall assume, as this second condition, that the coefficient of v in (4) be zero. Whence

$$(5) \qquad \frac{du}{dx} - \frac{2u}{x+1} = 0,$$

or

$$\frac{du}{u} = \frac{2\,dx}{x+1}.$$

Integrating, we find

$$\log u = 2 \log (x+1),$$

or

$$u = (x+1)^2.$$

Here we choose the constant of integration as zero. This is permissible, as the general solution of (5) is not needed. It is only necessary to add the constant after the final integration in the solution.

This value of u makes the transformed equation (4) reduce to

$$(x+1)^2\frac{dv}{dx} = (x+1)^3,$$

or

$$dv = (x+1)dx.$$

Whence

$$v = \frac{1}{2}(x+1)^2 + c.$$

Then finally

$$y = uv = \frac{1}{2}(x+1)^4 + c(x+1)^2.$$

2. Solve $ds + (s - 1) \tan t \, dt = 0$.

Solution. The equation may be written

$$\frac{ds}{dt} + s \tan t = \tan t.$$

Setting $s = uv$ gives

$$u \frac{dv}{dt} + \left(\frac{du}{dt} + u \tan t\right) v = \tan t.$$

From

$$\frac{du}{dt} + u \tan t = 0, \qquad \frac{du}{u} = - \tan t \, dt,$$

whence

$$\log u = \log \cos t, \qquad u = \cos t.$$

This value of u reduces the transformed equation to

$$\cos t \frac{dv}{dt} = \tan t, \qquad \text{or} \qquad dv = \tan t \sec t \, dt.$$

Therefore

$$v = \sec t + c.$$

The general solution is then

$$s = 1 + c \cos t.$$

222. Bernoulli's Equation. This equation may be written in the form

$$\frac{dy}{dx} + f_1(x) \cdot y = f_2(x) \cdot y^n,$$

and may be solved by the method given in the preceding article for the solution of the linear equation of the first order.

EXAMPLE

Solve $(y - xy^2)dx - dy = 0$.

Solution. This may be written

$$\frac{dy}{dx} - y = - xy^2. \tag{1}$$

Set $y = uv$, then $dy/dx = u \, dv/dx + v \, du/dx$ and we have

$$u \frac{dv}{dx} + \left(\frac{du}{dx} - u\right) v = - xu^2v^2. \tag{2}$$

By making the coefficient of v equal zero, we get

$$u = e^x.$$

This value of u in the equation (2) gives

$$e^x \frac{dv}{dx} = -xe^{2x}v^2, \qquad \text{or} \qquad \frac{dv}{v^2} = -xe^x dx.$$

Integrating, we have

$$-\frac{1}{v} = -(x-1)e^x + c, \qquad \text{or} \qquad v = \frac{1}{(x-1)e^x - c}.$$

Substituting in $y = uv$, we find the general solution to be

$$y = \frac{e^x}{(x-1)e^x - c}, \qquad \text{or} \qquad \frac{1}{y} = x - 1 - ce^{-x}.$$

223. Integrating Factors. Some equations which are not exact can be made so by multiplying the whole equation by some function. Such a multiplier is called an ***integrating factor.*** There is no convenient general method for finding integrating factors. A close inspection of the terms of the equation will sometimes lead to the proper factor. In so doing, certain differentials should be kept in mind. Thus

$$d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2},$$

$$d\left(\log\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{xy},$$

$$d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2 + y^2},$$

$$d\left(\sin^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y\sqrt{y^2 - x^2}},$$

and many others. The numerators of each of these given are the same and if it occurs in an equation, either $1/y^2$, $1/xy$, $1/x^2$, $1/(x^2 + y^2)$, or $1/y\sqrt{y^2 - x^2}$ may be the integrating factor needed.

Suppose the equation $M\,dx + N\,dy = 0$ is given and we wish to find whether or not there is an integrating factor $\phi(x)$, that is, a *function of x alone.* If there is such a $\phi(x)$,

$$\phi(x) \cdot M\,dx + \phi(x) \cdot N\,dy = 0$$

is exact; hence

$$\frac{\partial[\phi(x)\cdot M]}{\partial y}=\frac{\partial[\phi(x)\cdot N]}{\partial x},$$

or

$$\phi(x)\cdot\frac{\partial M}{\partial y}=\frac{d[\phi(x)]}{dx}\cdot N+\phi(x)\frac{\partial N}{\partial x}.$$

This may be written

$$\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]dx=\frac{d[\phi(x)]}{\phi(x)}.$$

This shows that there is such a function only if

$$\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]$$

is a *function of x alone.* If this is the case and if we call it $f(x)$, we have

$$f(x)dx=\frac{d[\phi(x)]}{\phi(x)},$$

whence $\phi(x)=e^{\int f(x)dx}$ is the integrating factor.

Similarly, there is an integrating factor which is a function of y alone if $[1/M][(\partial N)/(\partial x)-(\partial M)/(\partial y)]$ is a *function of y alone.* Calling this $f(y)$, the integrating factor is $e^{\int f(y)dy}$.

There are many other known integrating factors, but it is not necessary to include them here.

EXAMPLES

1. Solve by finding an integrating factor

$$x\,dy-y\,dx+x^2y\,dy+y^2x\,dx=0.$$

Solution. The terms $x\,dy-y\,dx$ suggest one of the factors mentioned at the beginning of this article. If we use $1/xy$ we have

$$\frac{dy}{y}-\frac{dx}{x}+x\,dy+y\,dx=0,$$

which is exact since $\partial M/\partial y = \partial N/\partial x$. Integrating, we have

$$\log y - \log x + xy = \log c,$$

or

$$y = cxe^{-xy}.$$

2. Integrate $x\,dy - y\,dx = (x^2 + y^2)^{1/2}dx$.
SOLUTION. If we divide through by the radical $(x^2 + y^2)^{1/2}$ we have

$$\frac{x\,dy - y\,dx}{\sqrt{x^2 + y^2}} = dx.$$

The first term of the numerator on the left suggests the need of x^2 as its denominator, so we have

$$\frac{\dfrac{x\,dy - y\,dx}{x^2}}{\dfrac{\sqrt{x^2 + y^2}}{x^2}} = dx.$$

This may be written

$$\frac{d\left(\dfrac{y}{x}\right)}{\sqrt{1 + \left(\dfrac{y}{x}\right)^2}} = \frac{dx}{x}.$$

Integrating, we get

$$\log\left[\frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}\right] = \log x + \log c,$$

therefore

$$y + \sqrt{x^2 + y^2} = cx^2.$$

PROBLEMS

Integrate each of the following equations. (Nos. 1–28.)

1. $x\,dy - y\,dx = x\,dx$. *Ans.* $y = x \log cx$.
2. $dy/dx + y = e^x$.
3. $dy - (xy + 2x)dx = 0$. *Ans.* $y = ce^{x^2/2} - 2$.
4. $(y - x)dy + y\,dx = 0$.
5. $3t\,ds + 6s\,dt = ts\,ds$. *Ans.* $t^2s = ce^{s/3}$.
6. $dy - 4x\,dx + xy\,dx = 0$.
7. $dx + (x \tan y - \sec y)dy = 0$. *Ans.* $x = \sin y + c \cos y$.
8. $x^4\,dy/dx + x^3(2x - 1)y = 1$.
9. $dy - 2xy\,dx + 4xy^2\,dx = 0$. *Ans.* $y(c + 2e^{x^2}) = e^{x^2}$.

10. $(v^2 + 1)^3 du + [4(v^2 + 1)^2 vu - 1]dv = 0.$

11. $xy^2 dy + y^3\,dx - y\,dx + 2\,y^2\,dx + x\,dy = 0.$
Ans. $xy - x/y + 2\,x = c.$

12. $(x + 1)dy/dx - 2\,y = e^x(x + 1)^3.$

13. $x(x\,dy + y\,dx) + y(x\,dy - y\,dx) + 2\,x^3 y \sin x^2\,dx = 3\,x^2 y^3\,dy.$
Ans. $\log xy + y/x - \cos x^2 = y^3 + c.$

14. $(x + 6\,xy\sqrt{x^2 + y^2})dx + (y + 3\,x^2\sqrt{x^2 + y^2})dy = 0.$

15. $x^2\,dy + (1 - 2\,x)y\,dx = x^2\,dx.$ *Ans.* $y = x^2(1 + ce^{1/x}).$

16. $(x^3y^2 - y)dx - (x^2y^3 + x)dy = 0.$

17. $x\,dy/dx + y + x^5y^4 = 0.$ *Ans.* $xy\sqrt[3]{3(c + x^2/2)} = 1.$

18. $dy - (xy - x)dx = 0.$

19. $dy/dx + xy = xy^2.$ *Ans.* $y(ce^{x^2/2} + 1) = 1.$

20. $dy + y \cos x\,dx = e^x\,dx.$

21. $(x + x^2y)dy = y\,dx.$ *Ans.* $x(c - y^2) = 2\,y.$

22. $x\,dx = y[\sec (x^2 + y^2) - 1]dy.$

23. $(2 - xy)y\,dx - (2 + xy)x\,dy = 0.$ *Ans.* $2 \log (x/y) - xy = c.$

24. $x\,(dy/dx) + y + x^4y^4e^x = 0.$

25. $2\,y\,dx + x\,dy - xye^{xy} \sin x\,dx + x^2ye^{xy} \cos x\,dy + xy^2e^{xy} \cos x\,dx = 0.$
Ans. $2 \log x + \log y + e^{xy} \cos x = c.$

26. $(\sqrt{x^2 - y^2} + x)dx + (\sqrt{x^2 - y^2} - y)dy = 0,$ and find the curve through the point (1, 0).

27. $(1 + x^2)dy/dx + y = \tan^{-1} x,$ and find the curve through the origin.
Ans. $y = \tan^{-1} x - 1 + e^{-\tan^{-1} x}.$

28. $dy/dx + y \cos x = (1/2) \sin 2\,x,$ and find the curve through $(\pi/2, 6).$

29. What family of curves cut the circles $x^2 + y^2 = a^2$ at 60°? Which of these curves passes through (1, 1)? *Ans.* $\log xy \pm \sqrt{3} \log (x/y) = 0.$

30. Find the orthogonal trajectories of $x^2 + y^2 - 2\,kx = a^2$, where a is fixed.

31. Solve $\sqrt{1 - x^2}\,dy/dx + y = \sin^{-1} x$ for the curve through (0, 1).
Ans. $y = \sin^{-1} x - 1 + 2\,e^{-\sin^{-1} x}.$

32. Solve a linear equation of your own choice or any one of this list and add a constant after the first integration. What happens to this constant in the final solution?

224. Clairaut Equations. If a differential equation can be arranged in the form

$$y = px + f(p), \tag{1}$$

where $p = dy/dx$, it is called a ***Clairaut equation.*** To solve such an equation, differentiate it with respect to x, so that

$$p = p + x\frac{dp}{dx} + \frac{d}{dp}[f(p)] \cdot \frac{dp}{dx}, \tag{2}$$

then

$$\left[x + \frac{d}{dp}\{f(p)\}\right]\frac{dp}{dx} = 0. \tag{3}$$

Now, discarding the bracket, which involves no differentials, we have

$$\frac{dp}{dx} = 0, \qquad \text{or} \qquad p = c.$$

Therefore

$$y = cx + f(c) \tag{4}$$

is the solution of (1).

225. Singular Solutions. It is interesting to note that the envelope of the particular solutions derived from equation (4) of the preceding article is also a solution of equation (1) of that article. This is true in general for differential equations; such solutions are called ***singular solutions,*** because they cannot be derived from the complete solutions by giving the constants of integration special values.

These singular solutions have little importance in elementary applications and so will not be discussed here.

PROBLEMS

Solve each of the following equations. In each case show that the envelope of the general solution is also a solution.

1. $y = px + 2p^2$. *Ans.* $y = cx + 2c^2$.

2. $y = px + e^{p^2}$.

3. $y = x\dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^3$. *Ans.* $y = cx - c^3$.

4. $y = x\dfrac{dy}{dx} + \tan^{-1}(dy/dx)$.

5. $2x^3\dfrac{dy}{dx} - 4x^2y = a\left(\dfrac{dy}{dx}\right)^2$, (Let $x^2 = u$). *Ans.* $y = cx^2 - ac^2$.

6. $x\dfrac{dy}{dx} + y = x^4\left(\dfrac{dy}{dx}\right)^2$, (Let $xy = u$).

7. $1 - 2y\dfrac{dx}{dy} = 4e^{2x}\left(\dfrac{dx}{dy}\right)^2$, (Let $e^{2x} = u$). *Ans.* $e^{2x} = cy + c^2$.

ADDITIONAL PROBLEMS

Integrate each of the following equations. (Nos. 1–26.)

1. $(1 + x)y^2\,dx - x^3\,dy = 0$. *Ans.* $2x^2 - 2xy - y = 2cx^2y$.

2. $(x - 1)dy - 2y\,dx = (x - 1)^5\,dx$.

3. $dy/dx + y/x = x^2$. *Ans.* $4xy = x^4 + c$.

4. $x\dfrac{dy}{dx} = y + \sqrt{x^2 - y^2}$.

5. $dy/dx - ay/x = (x + 1)/x$. *Ans.* $y = x/(1 - a) - 1/a + cx^a$.

6. $t\,ds - s\,dt = te^t\,dt$.

7. $x\,dy - y\,dx = x^2e^{2x}\,dx$. *Ans.* $2y = xe^{2x} + cx$.

8. $(1 + x^2)dy/dx + 2xy = 4x^2$.

9. $(x^2 + y^2 - a^2)x\,dx + (x^2 - y^2 - b^2)y\,dy = 0$.
Ans. $x^4 - y^4 - 2a^2x^2 - 2b^2y^2 + 2x^2y^2 = c$.

10. $e^y\,dx - y\sin y\,dy + xe^y\,dy = 0$.

11. $(x\sin xy - 1)(x\,dy + y\,dx) = y\,dy + \cos xy\,dx$.
Ans. $x\cos xy + xy + y^2/2 = c$.

12. $x\,dy - y\,dx = x^2y^2\sec(y/x)dy$.

13. $x\cos^2 y\,dx + \csc x\,dy = 0$. *Ans.* $\sin x + \tan y - x\cos x = c$.

14. $(vu - v^3)dv + du = 0$.

15. $dx + (x - y)dy = 0$. *Ans.* $x = y - 1 + ce^{-y}$.

16. $[(x - y)e^{y/x} + x]dx + xe^{y/x}dy = 0$.

17. $xy\dfrac{dy}{dx} = y^2 + 1$. *Ans.* $y = \sqrt{cx^2 - 1}$.

18. $y\dfrac{dp}{dy} + p - 2y = 0.$

19. $dy + 4xy\,dx = x^2dy.$ *Ans.* $(1 - x^2)^2 = cy.$

20. $(y^2 + x + 1)y\,dy - (y^2 + 1)dx = 0.$

21. $dy/dx = e^{x+y}.$ *Ans.* $ce^y + e^{x+y} + 1 = 0.$

22. $(y^2 - 4x^2)dx + xy\,dy = 0.$

23. $(y + x^2 - 1)x\,dx - (x^2 - 1)dy = 0.$ *Ans.* $y = x^2 - 1 + c\sqrt{x^2 - 1}.$

24. $(4x + y)dx + (x - 4y)dy = 0.$

25. $t\,ds - t^2s^3\,dt = s\,dt.$ *Ans.* $s\sqrt{c - t^4} = t\sqrt{2}.$

26. $v^2\,du = u^2\,dv - uv\,du.$

27. Solve the linear equation $dy/dx + f_1(x)\cdot y = f_2(x)$. This solution is often used as a formula.

$$Ans.\ y = e^{-\int f_1(x)dx}\left[\int \{f_2(x)\cdot e^{\int f_1(x)dx}\}dx + c\right].$$

28. Apply the formula derived in Problem 27 to solve Problems 3 and 15 of this group.

CHAPTER XXI

DIFFERENTIAL EQUATIONS OF HIGHER ORDER

226. Some Special Types of Higher Order. Certain equations of the second order which occur frequently in elementary applications will now be discussed.

$$\text{(I)} \qquad \frac{d^2y}{dx^2} = f(x), \qquad \text{or} \qquad \frac{d^ny}{dx^n} = f(x).$$

These equations are usually solved by repeated integration with respect to x. However, if we set $d^{n-1}y/dx^{n-1} = p$, the equation becomes of the first order.

EXAMPLE

Solve $\dfrac{d^3y}{dx^3} = xe^x$.

SOLUTION. One integration gives

$$\frac{d^2y}{dx^2} = (x - 1)e^x + c_1.$$

Integrating again, we have

$$\frac{dy}{dx} = (x - 1)e^x - e^x + c_1x + c_2$$

$$= (x - 2)e^x + c_1x + c_2.$$

The final integration gives the solution as

$$y = (x - 3)e^x + \frac{c_1x^2}{2} + c_2x + c_3.$$

$$\text{(II)} \qquad \frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right).$$

We reduce this equation to one of the first order by means of the substitution $dy/dx = p$. This substitution is made because the dependent variable is missing, and p is used as a dependent vari-

able, since we use for d^2y/dx^2 the value dp/dx. The resulting equation

$$\frac{dp}{dx} = f(x, p)$$

is of the first order in p and x.

EXAMPLE

Solve $\quad (1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0.$

SOLUTION. Set $p = dy/dx$, then $dp/dx = d^2y/dx^2$, and we have

$$(1 - x^2)\frac{dp}{dx} - px = 0,$$

in which the variables are separable. Hence, we write it in the form

$$\frac{dp}{p} = -\frac{x\,dx}{x^2 - 1}.$$

Integrating, we have

$$\log p = -\frac{1}{2}\log(x^2 - 1) + \log c_1,$$

or

$$p = c_1(x^2 - 1)^{-1/2}.$$

Therefore

$$dy = c_1\frac{dx}{\sqrt{x^2 - 1}},$$

and integrating we have the solution as

$$y = c_1 \log(x + \sqrt{x^2 - 1}) + c_2.$$

A special equation of this type occurs in § 237 of Chapter XXII, where the suspension bridge is treated. This same substitution leads there to a solution involving the hyperbolic cosine.

(III) $$\frac{d^2y}{dx^2} = f\left(y, \frac{dy}{dx}\right).$$

Again we reduce the order of the equation by a substitution. Setting $p = dy/dx$, we have

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy},$$

and we are able to remove x, which occurs only implicitly. The equation becomes

$$p\frac{dp}{dy} = f(y, p)$$

which is of the first order in p and y.

EXAMPLE

✓Solve $\quad y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = y^3.$

SOLUTION. Let $p = dy/dx$ and $p\,dp/dy = d^2y/dx^2$. Then we have

$$py\frac{dp}{dy} - p^2 = y^3,$$

or

$$\frac{dp}{dy} - \frac{p}{y} = \frac{y^2}{p},$$

which may be solved by the Bernoulli equation method presented in § 222. Thus let $p = uv$; then $dp/dy = u\,dv/dy + v\,du/dy$, and we have

$$u\left(\frac{dv}{dy} - \frac{v}{y}\right) + v\frac{du}{dy} = \frac{y^2}{uv}.$$

Obtaining v so that the coefficient of u is zero, we find that

$$v = y.$$

This makes the equation reduce to

$$u\,du = dy,$$

whence

$$u = \pm\sqrt{2y + 2c_1}.$$

But $p = dy/dx = uv$ and so

$$\frac{dy}{dx} = \pm y\sqrt{2y + 2c_1}.$$

Therefore

$$\frac{dy}{y\sqrt{2y + 2c_1}} = \pm dx,$$

and integration gives the solution as

$$\log\frac{\sqrt{y + c_1} - \sqrt{c_1}}{\sqrt{y + c_1} + \sqrt{c_1}} = \pm\sqrt{2c_1}(x + c_2).$$

(IV) $$\frac{d^2y}{dx^2} = f(y).$$

This equation is merely a special form of III but it is of sufficient importance to be solved separately. Since dy/dx is missing, we merely set $p\,dp/dy$ for d^2y/dx^2 and have

$$p\,dp = f(y)dy.$$

This gives for one integration

$$p^2 = 2\int f(y)dy + c,$$

a result in which the variables are separable.

This type of equation occurs in problems on motion in which the acceleration along the path is proportional to the distance traversed. Such an equation is solved as the example of this type.

EXAMPLE

Solve $d^2s/dt^2 = \pm k^2s$.

SOLUTION. Setting d^2s/dt^2 equal to $p\,dp/ds$, where p is ds/dt, this equation becomes

$$p\,dp = \pm k^2s\,ds.$$

Integration gives

$$p^2 = \left(\frac{ds}{dt}\right)^2 = \pm k^2(s^2 + c_1).$$

This is called the ***energy integral*** since $ds/dt = v$ and therefore, if we multiply the equation by $m/2$, it becomes

$$\frac{1}{2}mv^2 = \pm\frac{mk^2}{2}(s^2 + c_1).$$

That is, *the kinetic energy of a body of mass m and acceleration* $\pm k^2s$ *is given by*

$$\pm\frac{mk^2}{2}(s^2 + c_1).$$

This equation must be solved as two types, depending upon the sign before k^2.

(*a*) If the *sign before* k^2 *is positive*, the energy integral reduces to

$$\frac{ds}{dt} = \pm k\sqrt{s^2 + c_1};$$

whence

$$\frac{ds}{\sqrt{s^2 + c_1}} = \pm k\,dt.$$

Integrating, we have

$$\log\,(s + \sqrt{s^2 + c_1}) = \pm\, kt + \log c_2,$$

or, solving for s,

$$s = ae^{\pm kt} + be^{\mp kt},$$

where $a = c_2/2$ and $b = -\,c_1/2\,c_2$.

This solution may also be written in the form

$$s = A \sinh\,(\pm\, kt) + B \cosh\,(\pm\, kt),$$

where $B + A = 2\,a$ and $B - A = 2\,b$.

(*b*) If the *sign before* k^2 *is negative*, the constant c_1 must be negative and such that $|c_1| > s^2$; otherwise ds/dt would be imaginary. Writing c_1 as $(-\,a^2)$ the energy integral becomes

$$\frac{ds}{dt} = \pm\, k\sqrt{a^2 - s^2}.$$

Hence

$$\frac{ds}{\sqrt{a^2 - s^2}} = \pm\, k\, dt.$$

Integration gives

$$\sin^{-1}\frac{s}{a} = \pm\, kt + c,$$

or

$$\begin{aligned} s &= a \sin\,(\pm\, kt + c) \\ &= A \sin kt + B \cos kt, \end{aligned}$$

where $A = \pm\, a \cos c$ and $B = \pm\, a \sin c$.

We have seen this last equation as that which defines ***simple harmonic motion.*** Hence, *simple harmonic motion is the only possible motion in which the acceleration along the path is proportional to the distance from a fixed point, if the constant of proportionality is negative.* (See § 104.)

PROBLEMS

Solve each of the following equations. (Nos. 1–18.)

1. $d^2y/dx^2 = 1/x$. *Ans.* $y = x \log x - x + c_1 x + c_2$.

2. $d^2y/dx^2 = 1/\sqrt{1 - x^2}$.

3. $d^2y/dx^2 = \sec^2 ax$. *Ans.* $y = (1/a^2) \log \sec ax + c_1 x + c_2$.

4. $d^3y/dx^3 - x^2 = 0$.

5. $(1 - x^2)d^2y/dx^2 - x\, dy/dx = 0$. *Ans.* $y = c_1 \sin^{-1} x + c_2$.

6. $d^2y/dx^2 + dy/dx = (dy/dx)^2$.

7. $s\, d^2s/dt^2 + (ds/dt)^2 = 1$.
Ans. $\sqrt{s^2 \pm c_1^2} = t + c_2$, according as $ds/dt \gtrless 1$.

8. $d^2u/dv^2 = [1 + (du/dv)^2]^{1/2}$.

9. $(1 + x)d^2x/dy^2 + (dx/dy)^2 = 0$. *Ans.* $x^2/2 + x = c_1y + c_2$.

10. $(1 + v)d^2u/dv^2 - du/dv = 0$.

11. $y\, d^2y/dx^2 - (dy/dx)^2 + (dy/dx)^3 = 0$. *Ans.* $y - c_1 \log y = x + c_2$.

12. $d^2x/dy^2 - m^2x = 0$.

13. $d^2y/dx^2 + dy/dx = \sin 2\,x$.
Ans. $y = c_1 + c_2e^{-x} - (\cos 2\,x + 2 \sin 2\,x)/10$.

14. $y\, d^2y/dx^2 - (dy/dx)^2 = y^2 \log y$.

15. $d^2s/dt^2 - ds/dt = 3\,t$, if $s = 0$, and $ds/dt = 1$ at $t = 0$.
Ans. $s = 4(e^t - 1) - (3/2)t^2 - 3\,t$.

16. $dy/dx + \sqrt{x^2 + (dy/dx)^2} - x\, d^2y/dx^2 = 0$, if $y = 3$, and $dy/dx = 2$ at $x = 1$.

17. $2\,y\, d^2y/dx^2 + 2(dy/dx)^2 = y\, dy/dx$, if $x = 0$, $y = 2$, and $dy/dx = 3$.
Ans. $y^2 + 20 = 24\, e^{x/2}$.

18. $(1 - y)d^2y/dx^2 + (dy/dx)^2 = 0$, if $y = 3$, and $dy/dx = 2$ at $x = 1$.

19. Due to frictional resistance, the angular acceleration of a certain wheel is negative and is proportional to its angular velocity. If the wheel slows 30% in 20 sec., when will it slow 50%? *Ans.* 38.87 sec.

20. The angle θ made by the position of a pendulum with the vertical satisfies approximately the equation $d^2\theta/dt^2 = (-g/l)\theta$, where g and l are constants and the independent variable t is the time. Find θ in terms of t if the pendulum is released with zero velocity at $t = 0$ from the position $\theta = 0.1$ radian.

21. Show that the circle is the only curve for which the curvature at every point is a constant. (HINT: Take the curve in such a position that $y' = 0$ at $x = 0$.)

22. Solve $d^2y/dx^2 - m^2y = 0$ by the substitution $dy/dx = p$; now solve it again by using $y = e^{kx}$.

23. The acceleration of gravity varies inversely as the square of the distance from the center of the earth. With what velocity must a body be projected from the surface of the earth to escape? (Use as radius of the earth $21 \cdot 10^6$ ft.)
Ans. 7 miles per second.

24. Prove that a particle sliding without friction from the highest point of a circle in the vertical plane along any chord will reach the circumference in the same time it would fall along the vertical diameter. (HINT: Acceleration along a chord with inclination of θ is $d^2s/dt^2 = g \sin \theta$.)

227. Linear Equations with Constant Coefficients. Linear differential equations with constant coefficients are of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x),$$

where $a_0, a_1, a_2, \cdots, a_n$ are constants.

If $f(x) = 0$, the equation is called ***homogeneous;*** and if $f(x) \neq 0$, it is said to be ***complete.*** In this treatment, we shall limit the discussion almost entirely to equations not higher than the second order, as such equations occur most frequently in the elementary applications.

THEOREM. *If $y = u$ and $y = v$ are solutions of the homogeneous equation*

$$a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0, \tag{1}$$

then $y = c_1 u + c_2 v$, where c_1 and c_2 are independent of x, is also a solution.

PROOF. If $y = u$ is a solution of (1), on substituting u for y, we have

$$a_0 \frac{d^2 u}{dx^2} + a_1 \frac{du}{dx} + a^2 u = 0, \tag{2}$$

which can be written

$$a_0 \frac{d^2 (c_1 u)}{dx^2} + a_1 \frac{d(c_1 u)}{dx} + a_2 (c_1 u) = 0, \tag{3}$$

since c_1 is independent of x. In the same manner we get

$$a_0 \frac{d^2 (c_2 v)}{dx^2} + a_1 \frac{d(c_2 v)}{dx} + a_2 (c_2 v) = 0. \tag{4}$$

The addition of corresponding members of (3) and (4) gives

$$a_0 \frac{d^2 (c_1 u + c_2 v)}{dx^2} + a_1 \frac{d(c_1 u + c_2 v)}{dx} + a_2 (c_1 u + c_2 v) = 0, \tag{5}$$

which shows that $y = c_1u + c_2v$ is a solution of equation (1). This is moreover the ***general solution*** since it contains two arbitrary constants.

This method of proof evidently may be used for the corresponding theorem for an equation of the n-th order.

228. Solution of the Homogeneous Equation of the Second Order. A very simple method of solution is to set $y = uv$ and carry through operations similar to those for the solution of linear equations of the first order. The one difference is that here we determine u so that the coefficient of dv/dx is zero instead of the coefficient of v.

EXAMPLES

1. Solve $d^2y/dx^2 + 3\,dy/dx - 4\,y = 0$.

SOLUTION. Set $y = uv$, then $dy/dx = u\,dv/dx + v\,du/dx$, and therefore $d^2y/dx^2 = u\,d^2v/dx^2 + 2(du/dx)(dv/dx) + v\,d^2u/dx^2$. Substituting these in the equation, we get

$$\left(\frac{d^2u}{dx^2} + 3\frac{du}{dx} - 4\,u\right)v + \left(2\frac{du}{dx} + 3\,u\right)\frac{dv}{dx} + u\frac{d^2v}{dx^2} = 0.$$

Since the coefficient of v is in form the same as the left side of the original equation, we would not gain anything by imposing the condition that this coefficient vanish in order to determine u. Hence we turn to the next coefficient and determine u so that the coefficient of dv/dx is zero. Therefore we let

$$2\frac{du}{dx} + 3\,u = 0,$$

or

$$\frac{du}{u} = -\frac{3}{2}\,dx.$$

Whence

$$\log u = -\frac{3\,x}{2},$$

or

$$u = e^{-3x/2}.$$

As previously, we take the constant of integration as zero since there are still two integrations to perform. This value of u changes the uv equation into the form

$$\left[\frac{9}{4}\,e^{-3x/2} - \frac{9}{2}\,e^{-3x/2} - 4\,e^{-3x/2}\right]v + e^{-3x/2}\frac{d^2v}{dx^2} = 0.$$

Collecting and dividing out the factor $e^{-3x/2}$, which is not zero, we get

$$\frac{d^2v}{dx^2} = \frac{25\,v}{4},$$

which has been solved in § 226, **IV.** As formerly, we let $d^2v/dx^2 = p\,dp/dv$ and get

$$p\,dp = \frac{25\,v}{4}\,dv.$$

Integration gives

$$p^2 = \frac{25}{4}\,(v^2 + c_1).$$

Therefore, since $p = dv/dx$,

$$\frac{dv}{\sqrt{v^2 + c_1}} = \pm \frac{5}{2}\,dx,$$

and, integrating, we have

$$\log\,(v + \sqrt{v^2 + c_1}) = \pm \frac{5\,x}{2} + \log c_2,$$

whence

$$v + \sqrt{v^2 + c_1} = c_2 e^{\pm 5x/2},$$

and solving for v,

$$v = k_1 e^{\pm 5x/2} + k_2 e^{\mp 5x/2},$$

where $k_1 = c_2/2$, $k_2 = -c_1/2\,c_2$. Hence

$$\begin{aligned} y &= uv = e^{-3x/2}(k_1 e^{\pm 5x/2} + k_2 e^{\mp 5x/2}) \\ &= k_1 e^{x} + k_2 e^{-4x} \end{aligned}$$

is the general solution if the positive sign is taken for $5\,x/2$. The negative sign merely interchanges k_1 and k_2, which are the arbitrary constants.

2. Solve $d^2y/dx^2 - 4\,dy/dx + 4\,y = 0$.

SOLUTION. Set $y = uv$. Substituting in the given equation, we have

$$\left(\frac{d^2u}{dx^2} - 4\,\frac{du}{dx} + 4\,u\right) v + \left(2\,\frac{du}{dx} - 4\,u\right) \frac{dv}{dx} + u\,\frac{d^2v}{dx^2} = 0.$$

As in the previous example, obtain u so that the coefficient of dv/dx is zero. That is, set

$$2\,\frac{du}{dx} - 4\,u = 0.$$

Then

$$u = e^{2x},$$

and substitution gives

$$\frac{d^2v}{dx^2} = 0.$$

Therefore

$$v = c_1x + c_2.$$

The general solution is then

$$y = (c_1x + c_2)e^{2x}.$$

3. Solve $d^2y/dx^2 + dy/dx + 2y = 0$.

SOLUTION. Set $y = uv$ and the equation becomes

$$\left(\frac{d^2u}{dx^2} + \frac{du}{dx} + 2u\right)v + \left(2\frac{du}{dx} + u\right)\frac{dv}{dx} + u\frac{d^2v}{dx^2} = 0.$$

Obtaining u so that

$$2\frac{du}{dx} + u = 0,$$

we have

$$u = e^{-x/2},$$

This reduces the equation above to

$$\frac{d^2v}{dx^2} = -\frac{7v}{4},$$

which we have solved previously.

Setting $d^2v/dx^2 = p\,dp/dv$ and integrating, we have

$$p^2 = \frac{7}{4}(c_1^2 - v^2).$$

Therefore, since $p = dv/dx$,

$$\frac{dv}{\sqrt{c_1^2 - v^2}} = \pm\frac{\sqrt{7}}{2}\,dx,$$

and

$$\sin^{-1}\frac{v}{c_1} = \frac{\pm x\sqrt{7} + c_2}{2}.$$

Whence

$$\begin{aligned} v &= c_1 \sin\frac{\pm x\sqrt{7} + c_2}{2} \\ &= k_1 \sin\left(\pm\frac{x\sqrt{7}}{2}\right) + k_2\cos\left(\pm\frac{x\sqrt{7}}{2}\right), \end{aligned}$$

where $k_1 = c_1 \cos (c_2/2)$ and $k_2 = c_1 \sin (c_2/2)$. Then the solution is

$$y = e^{-x/2}\left[k_1 \sin\left(\pm\frac{x\sqrt{7}}{2}\right) + k_2 \cos\left(\pm\frac{x\sqrt{7}}{2}\right)\right].$$

These three examples illustrate the only three possible forms which may arise from the integration of the second order linear equation with constant coefficients. That is, the value of d^2v/dx^2 is either 0 or $\pm k^2v$.

If, in the equation $ay'' + by' + cy = 0$, we substitute e^{mx} for y we obtain $e^{mx}(am^2 + bm + c) = 0$. Hence $y = e^{mx}$ will be a solution only if m is a root of the equation

$$(1) \qquad am^2 + bm + c = 0.$$

If these roots are real and distinct, say m_1 and m_2, then $y = e^{m_1x}$ and $y = e^{m_2x}$ are particular solutions and the general solution is $y = c_1e^{m_1x} + c_2e^{m_2x}$. Compare Example 1.

If m_1 is a double root of (1), we would obtain thus only one particular solution. However, as in Example 2, the u in the substitution $y = uv$ becomes the particular solution e^{m_1x} and as a result d^2v/dx^2 becomes zero. Hence the substitution $y = e^{m_1x}\cdot v$ where $v = c_1x + c_2$ will give the general solution $y = e^{m_1x}(c_1x + c_2)$.

If (1) has complex roots, the substitution $y = e^{mx}$ is possible, and the general solution obtained may be changed by means of (9) in § 204 into a form similar to that in Example 3.

229. The Complete Equation of the Second Order. We shall give several methods for solving the complete equation with constant coefficients. The first given, while not general, is one that is readily applicable to the types most frequently met with in elementary applications. This method requires that we prove the theorem given below:

THEOREM. *The general solution of a complete linear equation with constant coefficients of the form*

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x)$$

is given by $y = w + z$ where $y = w$ is the general solution of the homogeneous equation

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0$$

*and $y = z$ is **any** particular solution of the complete equation.*

PROOF. Since $y = w$ is a solution of the homogeneous equation, we have

$$a_0 \frac{d^2w}{dx^2} + a_1 \frac{dw}{dx} + a_2 w = 0.$$

Also, by hypothesis, $y = z$ is a solution of the complete equation; therefore

$$a_0 \frac{d^2z}{dx^2} + a_1 \frac{dz}{dx} + a_2 z = f(x).$$

Adding the corresponding members of these two equations, we have

$$a_0 \frac{d^2(w+z)}{dx^2} + a_1 \frac{d(w+z)}{dx} + a_2(w+z) = f(x),$$

which shows that $y = w + z$ is a solution of the complete equation. It is the ***general solution*** because w contains two arbitrary constants.

The solution w of the homogeneous equation is called the ***complementary function*** of the complete equation.

This method of proof applies readily to equations of higher order.

230. A Method for Finding a Particular Integral. As our solutions of the homogeneous equation of § 228 are general and give us the complementary function for any complete equation, we need only a method of finding a particular integral of the complete equation. We give here the method of ***undetermined coefficients.*** Although the method is not general, it is sufficient in all cases where the right-hand member $f(x)$ contains only such terms as have a finite number of distinct derivatives; that is, terms like x^n (n a positive integer), e^{ax}, $\sin ax$, $\cos ax$, constants, and products of any of these. The form of $f(x)$ is most frequently of this type in the equations which arise from elementary applications to physics and engineering. The following examples illustrate the method.

EXAMPLES

1. Solve $d^2y/dx^2 + dy/dx - 6y = \sin x$.

SOLUTION. The complementary function is found as in § 228 to be

$$y_0 = k_1 e^{-3x} + k_2 e^{2x}.$$

To find a particular solution, we examine the right-hand member. This suggests that there may be a solution of the form

$$y_1 = a \sin x.$$

On substituting this value for y on the left-hand side, we get

$$-7a \sin x + a \cos x = \sin x.$$

Since this equation must be true for all values of x, we may equate the corresponding coefficients to determine the constant a. But this gives

$$-7a = 1, \qquad a = 0,$$

which are not consistent equations. Hence $y = a \sin x$ is not a particular solution.

Since derivatives of y appear on the left-hand side, suppose we try as a *value for y_1 the sum of such terms as appear in the right-hand member together with all such terms as can be derived from them by differentiation, each term having an undetermined coefficient.* In this example $\sin x$ and $\cos x$ are the only possible terms; therefore let

$$y_1 = a \sin x + b \cos x.$$

Substituting this in the complete equation, we get

$$(a - 7b) \cos x - (b + 7a) \sin x = \sin x.$$

Equating coefficients, we have

$$a - 7b = 0, \qquad -b - 7a = 1,$$

which determine a and b. Solving them, we obtain

$$a = -\frac{7}{50}, \qquad b = -\frac{1}{50}.$$

Therefore a particular solution is

$$y_1 = -\frac{7 \sin x}{50} - \frac{\cos x}{50}.$$

The general solution is

$$y = y_0 + y_1 = k_1 e^{-3x} + k_2 e^{2x} - \frac{7 \sin x}{50} - \frac{\cos x}{50}.$$

2. Solve $d^2y/dx^2 - 2\,dy/dx + 2y = x^2 + \sin 2x$.

Solution. The complementary function may be found, as previously explained, to be

$$y_0 = e^x(k_1 \sin x + k_2 \cos x).$$

The terms in the right-hand member are x^2 and $\sin 2x$. Their derivatives are then x, a constant, and $\cos 2x$, except for coefficients. Therefore we set

$$y_1 = a \sin 2x + b \cos 2x + cx^2 + dx + e.$$

Substituting this in the equation, we get

$$(4b - 2a) \sin 2x - (4a + 2b) \cos 2x + 2cx^2 + (2d - 4c)x + (2e - 2d + 2c) = x^2 + \sin 2x.$$

Equating corresponding coefficients, we have

$$4b - 2a = 1,\ 4a + 2b = 0,\ 2c = 1,\ 2d - 4c = 0,\ 2e - 2d + 2c = 0,$$

which are sufficient to determine the unknown constants if the equations are consistent. Solving, we have $a = -1/10$, $b = 1/5$, $c = 1/2$, $d = 1$, $e = 1/2$, and therefore $y_1 = -(\sin 2x)/10 + (\cos 2x)/5 + x^2/2 + x + 1/2$ is a particular solution of the given equation. The general solution then is $y = y_0 + y_1$ or

$$y = e^x(k_1 \sin x + k_2 \cos x) - \frac{\sin 2x}{10} + \frac{\cos 2x}{5} + \frac{x^2}{2} + x + \frac{1}{2}.$$

3. Find the general solution of $d^2y/dx^2 + dy/dx - 6y = x + e^{2x}$.

SOLUTION. The complementary function was found in the first example to be $y_0 = k_1e^{-3x} + k_2e^{2x}$. We *notice that a term like one of those in this complementary function is also found in the right-hand member.* The method of choosing y_1 as given above fails to give a particular integral when a term in the right-hand member is also a term in the complementary function. The method merely leads to inconsistent equations in the undetermined coefficients. For that reason we need the following suggestion. *If a term in the right-hand member is a term in the complementary function, replace that term and its derivatives by x^n times each, choosing $n = 1, 2, \cdots$ until the method of undetermined coefficients leads to consistent equations.* Thus, to solve the equation above for a particular integral, we select the following terms:

$$y_1 = axe^{2x} + cx + d.$$

Substituting this in the equation, we get

$$5ae^{2x} - 6cx + (c - 6d) = e^{2x} + x.$$

Therefore we have

$$5a = 1, \qquad -6c = 1, \qquad c - 6d = 0,$$

which give

$$a = \frac{1}{5}, \qquad c = -\frac{1}{6}, \qquad d = -\frac{1}{36}.$$

The general solution is therefore

$$y = k_1 e^{-3x} + k_2 e^{2x} + \frac{xe^{2x}}{5} - \frac{x}{6} - \frac{1}{36}.$$

231. Variation of Parameters. Another method of obtaining the general solution of the complete equation is to consider the constants in the complementary function as undetermined functions of the independent variable. Since there are n of the constants and the only condition on them is that the complementary function must satisfy the homogeneous equation, there are $n - 1$ further conditions which may be imposed. The conditions are imposed in such a manner as to simplify the work as the solution proceeds. The method as applied to a second-order equation will now be given. The same is readily seen to apply for any order. It may also be used if n particular solutions are known, even if the equation does not have constant coefficients.

EXAMPLE

Solve by variation of parameters the equation

$$\frac{d^2y}{dx^2} + y = \sec^2 x.$$

SOLUTION. This is evidently an example where the method of undetermined coefficients will not work, as $\sec^2 x$ has an infinite number of distinct derivatives. However, we find the complementary function to be

$$y_0 = c_1 \sin x + c_2 \cos x.$$

We now think of c_1 and c_2 as *parameters* which make y_0 a solution of the complete equation. This leaves one additional condition which may be imposed on the parameters. Differentiating y_0, we have

$$\frac{dy_0}{dx} = c_1 \cos x - c_2 \sin x + \sin x \frac{dc_1}{dx} + \cos x \frac{dc_2}{dx}.$$

Now, as the additional condition on c_1 and c_2, let us assume that the sum of the last two terms of dy_0/dx be equal to zero. That is, let

$$\sin x \frac{dc_1}{dx} + \cos x \frac{dc_2}{dx} = 0. \tag{1}$$

Then the second derivative of y_0 is

$$\frac{d^2y_0}{dx^2} = -c_1 \sin x - c_2 \cos x + \cos x \frac{dc_1}{dx} - \sin x \frac{dc_2}{dx}.$$

Substituting for y_0 and d^2y_0/dx^2 in the original equation and simplifying, we have

$$(2) \qquad \cos x \frac{dc_1}{dx} - \sin x \frac{dc_2}{dx} = \sec^2 x.$$

Solving (1) and (2) simultaneously for dc_1/dx and dc_2/dx, we find

$$\frac{dc_1}{dx} = \sec x, \qquad \frac{dc_2}{dx} = -\sec x \tan x.$$

Therefore

$$c_1 = \log (\sec x + \tan x) + k_1, \qquad c_2 = -\sec x + k_2.$$

Substituting these values in y_0, we find the general solution in the form

$$y = k_1 \sin x + k_2 \cos x + \sin x \log (\sec x + \tan x) - 1.$$

232. A General Method for Solving the Complete Equation of the Second Order. Consider the equation

$$(1) \qquad a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = f(x).$$

If we set $y = uv$, this becomes

$$(2) \quad \left(a_0 \frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_2 u\right) v + \left(2a_0 \frac{du}{dx} + a_1 u\right) \frac{dv}{dx} + a_0 u \frac{d^2v}{dx^2} = f(x).$$

Now suppose we impose the condition on u that shall make the coefficient of v in (2) zero. That is, let

$$(3) \qquad a_0 \frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_2 u = 0.$$

This equation is readily solved for u since it is the homogeneous equation solved in § 228. From (3) we determine a particular value for u, say $u = \phi(x)$.

Using this value of u, equation (2) reduces to

$$(4) \qquad \left\{2a_0 \frac{d}{dx} [\phi(x)] + a_1\phi(x) \right\} \frac{dv}{dx} + a_0\phi(x) \frac{d^2v}{dx^2} = f(x),$$

where $\phi(x)$ and $f(x)$ are both known functions. Now (4) is of form II of § 226, and writing $p = dv/dx$ and $dp/dx = d^2v/dx^2$, this reduces to a linear equation of the first order and the solution is readily completed.

EXAMPLES

1. Solve $d^2y/dx^2 + 2\,dy/dx + y = x$.

SOLUTION. Setting $y = uv$ and determining u so that the coefficient of v in the transformed equation is zero, we have

$$u = e^{-x}.$$

This value of u makes the transformed equation reduce to

$$\frac{d^2v}{dx^2} = xe^x.$$

Therefore, integrating this twice, we have

$$v = xe^x - 2\,e^x + c_1x + c_2,$$

whence the complete solution is given immediately by

$$y = uv = x - 2 + (c_1x + c_2)e^{-x}.$$

2. Solve $d^2y/dx^2 + dy/dx - 6\,y = x + e^{2x}$.

SOLUTION. If y is set equal to uv, this equation is transformed into the following:

$$\left(\frac{d^2u}{dx^2} + \frac{du}{dx} - 6\,u\right)v + \left(2\,\frac{du}{dx} + u\right)\frac{dv}{dx} + u\,\frac{d^2v}{dx^2} = x + e^{2x}.$$

If u makes the coefficient of v zero, we have as previously $u = e^{2x}$. This value of u makes the equation above reduce to

$$\frac{d^2v}{dx^2} + 5\,\frac{dv}{dx} = xe^{-2x} + 1.$$

Now let $dv/dx = p$ and $d^2v/dx^2 = dp/dx$ and

$$\frac{dp}{dx} + 5\,p = xe^{-2x} + 1.$$

This is a linear equation of the first order in p and so is readily solved as in § 221. The solution is

$$p = \frac{xe^{-2x}}{3} - \frac{e^{-2x}}{9} + \frac{1}{5} + c_1e^{-5x}.$$

Integrating, we have

$$v = -\frac{xe^{-2x}}{6} - \frac{e^{-2x}}{36} + \frac{x}{5} - \frac{c_1e^{-5x}}{5} + c_2,$$

whence the general solution $y = uv$ is

$$y = \frac{xe^{2x}}{5} - \frac{x}{6} - \frac{1}{36} + k_1e^{2x} + k_2e^{-3x}.$$

233. Systems of Linear Equations with Constant Coefficients. If the equations have only one independent variable and as many dependent variables as there are equations, we proceed as follows.

Differentiate the given equations until the original and derived equations are sufficient for eliminating all but one of the dependent variables and their derivatives. We shall limit our discussion to two first-order equations so as to make the derived equation not higher than the second order. Systems of higher order are treated in the same manner.

EXAMPLE

Solve the system

$$\begin{cases} \dfrac{dy}{dt} + x = \sin t, \\[2ex] \dfrac{dx}{dt} + y = \cos t. \end{cases}$$

SOLUTION. This system has only dy/dt and y to eliminate, and as we need $n + 1$ equations to eliminate n quantities, it is necessary to differentiate one of the equations to get a third equation. We use the second because differentiation does not introduce a higher derivative of y than already exists in the equations. This gives

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} = -\sin t.$$

Eliminating dy/dt and y between the three equations, we get

$$\frac{d^2x}{dt^2} - x = -2\sin t,$$

which is a linear equation with constant coefficients. Solving this equation, we have x as a function of t with two constants of integration. Substituting this value of x in the second equation, we get y at once.

The method may be applied to equations of higher order if repeated differentiation is used. Enough equations can always be found because in general only one new derivative is introduced when two equations are differentiated.

PROBLEMS

Solve the following equations.

1. $d^2y/dx^2 - 7\,dy/dx + 12\,y = 0.$ *Ans.* $y = c_1e^{4x} + c_2e^{3x}.$

2. $d^2y/dx^2 + 4\,y = 0.$

3. $d^2y/dx^2 + 6\,dy/dx + 9\,y = 0.$ *Ans.* $y = e^{-3x}(c_1x + c_2).$

4. $d^2y/dx^2 - dy/dx + y = 0.$

5. $d^2s/dt^2 - 3\,ds/dt = 2 - 6\,t.$ *Ans.* $s = c_1 + c_2e^{3t} + t^2.$

6. $d^2u/dv^2 - 7\,du/dv + 10\,u = 3\,ve^{2v}.$

7. $d^2\theta/dt^2 + 4\,d\theta/dt + 4\,\theta = \cos t.$
Ans. $\theta = e^{-2t}(c_1t + c_2) + (3\cos t + 4\sin t)/25.$

8. $d^2y/dx^2 + dy/dx - 6\,y = \sin 2\,x.$

9. $d^2x/dy^2 - 7\,dx/dy + 12\,x = e^{3y}.$ *Ans.* $x = c_1e^{4y} + c_2e^{3y} - ye^{3y}.$

10. $d^2y/dx^2 + 6\,dy/dx + 9\,y = e^x \sin x.$

11. $d^2y/dx^2 + 4\,dy/dx + 13\,y = \sin x.$
Ans. $y = (c_1 \cos 3\,x + c_2 \sin 3\,x)e^{-2x} + (3\sin x - \cos x)/40.$

12. $d^2y/dx^2 - 2\,dy/dx + y = x + e^{2x}.$

13. $d^2x/dy^2 - dx/dy + x = y - 1.$
Ans. $x = e^{y/2}[c_1 \cos (1/2)y\sqrt{3} + c_2 \sin (1/2)y\sqrt{3}] + y.$

14. $d^2y/dx^2 + 9\,y = 2\sin 3\,x.$

15. $d^2s/dt^2 - ds/dt - 2\,s = e^{-t}.$ *Ans.* $s = c_1e^{2t} + c_2e^{-t} - te^{-t}/3.$

16. $d^2y/dx^2 + 4\,y = \tan 2\,x.$

17. $d^2x/dy^2 - 5\,dx/dy + 6\,x = 3\,e^{3y}.$ *Ans.* $x = c_1e^{2y} + c_2e^{3y} + 3\,ye^{3y}.$

18. $d^2y/dx^2 + 4\,y = \csc x.$

19. $d^2y/dt^2 + y = 3\sec^2 t.$
Ans. $y = c_1 \sin t + c_2 \cos t$ where $c_1 = 3\log(\sec t + \tan t) + k_1,$
$c_2 = k_2 - 3\sec t.$

20. $d^2y/dx^2 - 2\,dy/dx + y = e^x/x.$

21. $\begin{cases} dy/dt - x = 2, \\ dy/dt + dx/dt = 2. \end{cases}$ *Ans.* $x = c_1e^{-t},\ y = c_2 + 2\,t - c_1e^{-t}.$

22. $\begin{cases} dy/dt + x = \cos t, \\ dx/dt + y = \sin t. \end{cases}$

23. $\begin{cases} dy/dt + x = 0, \\ dx/dt + 3\,x + 4\,y = 0. \end{cases}$ *Ans.* $x = 4\,c_2e^{-4t} - c_1e^t,\ y = c_1e^t + c_2e^{-4t}.$

24. $\begin{cases} dy/dt + 3\,x = e^{2t}, \\ dx/dt + 3\,y = e^{-2t}. \end{cases}$

25. $d^2y/dx^2 - 5\,dy/dx + 6\,y = x + e^x$, if $y = 0$, $dy/dx = 1$ at $x = 0$.
Ans. $y = (58e^{3x} - 81\,e^{2x} + 18e^x + 6\,x + 5)/36.$

26. $d^2y/dx^2 - dy/dx - 2\,y = e^x + e^{-x}$, if $y = 1$, $dy/dx = 1/6$ at $x = 0$.

27. $d^2y/dx^2 + 6\,dy/dx + 5\,y = e^{2x}$, if $y = 1/6$, $dy/dx = -1/6$ at $x = 0$.
Ans. $y = (3e^{-5x} + 7e^{-x} + 4e^{2x})/84.$

28. $d^2s/dt^2 + a^2s = \cos at$, if $s = 0$, $ds/dt = 1$ at $t = 0$.

29. $d^2s/dt^2 - 6\,ds/dt - 16\,s = \sin t$, if $s = 0$, $ds/dt = 0$ at $t = 0$.

Ans. $s = (e^{8t} - 13e^{-2t} + 12\cos t - 34\sin t)/650$.

30. $d^3y/dx^3 + 3\,d^2y/dx^2 + 3\,dy/dx + y = 5\,e^x \sin x$.

234. An Approximate Graphical Solution of the First-Order Equation. The first-order equation may be written in the form

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

which has as its general solution a *one-parameter family of curves.* To determine approximately the graph of one curve of the family we take $P_1(x_1, y_1)$ as a point of the curve. The coordinates of P_1 substituted in (1) give dy_1/dx_1 or the slope of the curve at the point P_1. Through P_1 draw a line-segment with this slope; then take a convenient Δx from P_1 and construct the resulting Δy to the line through P_1. This locates a point P_2 near P_1. Find the slope at P_2 by substituting x_2 and y_2 in (1) and continue the operations. *The smooth curve through the points so determined is an approximate solution.* The solution is more nearly correct, the smaller Δx is taken. This is evident because each point after the first is located on the tangent to the curve through the previous point. Also it is evident that large values of dy/dx are not conducive to a good approximation. If $f(x, y)$ is n-valued there are n curves.

EXAMPLE

Find graphically the approximate solution of $dy/dx - xy = 0$ which passes through the point (1, 1).

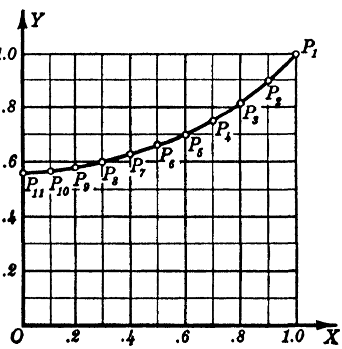


FIG. 228

SOLUTION. Suppose we use $\Delta x = -0.1$ and thereby find the following table and resulting figure.

P_1: (1, 1),	$m_1 = 1.$
P_2: (0.9, 0.9),	$m_2 = 0.81.$
P_3: (0.8, 0.82),	$m_3 = 0.66.$
P_4: (0.7, 0.75),	$m_4 = 0.53.$
P_5: (0.6, 0.7),	$m_5 = 0.42.$
P_6: (0.5, 0.66),	$m_6 = 0.33.$
P_7: (0.4, 0.63),	$m_7 = 0.25.$
P_8: (0.3, 0.6),	$m_8 = 0.18.$
P_9: (0.2, 0.58),	$m_9 = 0.12.$
P_{10}: (0.1, 0.57),	$m_{10} = 0.06.$
P_{11}: (0, 0.56).	

The accuracy of this may be observed from the fact that the ordinary solution would locate P_{11} at (0, 0.61).

235. An Approximate Graphical Solution of the First Order Equation by Means of Trajectories. In the equation $f(x, y, dy/dx) = 0$, substitute in succession $dy/dx = c_1, c_2, c_3, \cdots$ and plot the resulting curves. Across each curve draw a system of parallel ***lineal elements*** (short line-segments) with their slope equal to the value of dy/dx which determined the curve. These lineal elements give the directions of the integral curves as they cross each of the curves determined by the values $c_1, c_2, c_3, \cdots$.

If we start at a point on any curve and go in the direction of the lineal elements that cross it and approach each successive curve in the direction of the elements across it, we can draw approximate solutions of the equation. Of course the values c_i should be at most consecutive integers in order to have enough trajectories for an accurate system of solutions.

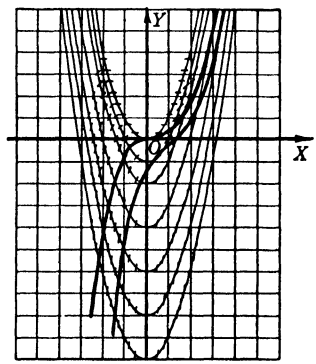


FIG. 229

EXAMPLE

Solve by trajectories $x^2 - y - dy/dx = 0$.

SOLUTION. Let $dy/dx = 0, 1, 2, 4, 6, 8, 10$. The resulting equations give the parabolas shown in Fig. 229. The lineal elements show the value of dy/dx for each parabola. The heavy curves are approximate solutions of the equation.

236. Approximate Graphical Solution of Equations of the Second Order. This equation may be written

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right). \tag{1}$$

If we substitute x_1, y_1, and dy_1/dx_1 in equation (1) and solve the resulting equation for d^2y_1/dx_1^2, its sign tells us whether the integral curve is concave upward or downward at the point $P_1(x_1, y_1)$. In doing this we select a point and assign a slope m_1 at the point and thus determine the value of d^2y/dx^2 for these assumed values. Then dy_1/dx_1 and d^2y_1/dx_1^2 allow us to find the radius of curvature R_1 at P_1. Whence we may lay off R_1 along the line through P_1 with slope $-1/m_1$, above P_1 if $d^2y_1/dx_1^2 > 0$, below P_1 if $d^2y_1/dx_1^2 < 0$. Now use R_1 as a

radius to construct a small arc P_1P_2 of the circle of curvature of the curve at P_1. Then find from the graph the coordinates of P_2 and the slope of the arc P_1P_2 at P_2. Using these coordinates and this slope, repeat the operations. The smooth curve through the points P_i is the desired approximate solution. The points P_i should be located with the help of the graph paper or even more accurately by use of trigonometry. Moreover, the arcs should be kept very short. If $f(x, y, dy/dx)$ is n-valued there are n curves.

EXAMPLE

Obtain a graphical solution of the differential equation

$$y^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0.$$

SOLUTION. Take as P_1 the point (0, 5) and let $m_1 = 0$. These give the following table and figure (Fig. 230):

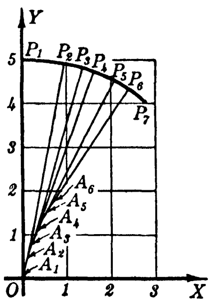


FIG. 230

P_1: (0, 5), $m_1 = 0$, $R_1 = -5$.

P_2: (0.95, 4.9), $m_2 = -\frac{5}{26}$, $R_2 = -4.5$.

P_3: (1.35, 4.8), $m_3 = -\frac{5}{17}$, $R_3 = -4.07$.

P_4: (1.6, 4.7), $m_4 = -\frac{5}{14}$, $R_4 = -3.78$.

P_5: (2.05, 4.5), $m_5 = -\frac{1}{2}$, $R_5 = -3.28$.

P_6: (2.4, 4.3), $m_6 = -\frac{2}{3}$, $R_6 = -3.0$.

P_7: (2.8, 4.0).

PROBLEMS

Solve graphically each of the following equations.

1. $xy - x^3 + dy/dx = 0$, $(-1, 0)$.

2. $x\,dy/dx - y^2 = 0$, $(1, 1)$.

3. $y\,dy/dx + x = 0$, $(0, 2)$.

4. $x\,dy/dx - y\sin x = 0$, $(1, 1.5)$.

5. $x^2\, dy/dx + y = 0$, by trajectories.

6. $x^2 - y^2 - dy/dx = 0$, by trajectories.

7. $x\, d^2y/dx^2 - x^2\, dy/dx - y = 0, \quad (1, 1), \quad m_1 = 2.$

8. $(1 + y)d^2y/dx^2 + (dy/dx)^2 = 3\,x - 10, \quad (1/2, 1/2), \quad m_1 = 5.$

9. $y^2\, d^2y/dx^2 - 4\,x\, dy/dx + y = 0, \quad (0, 3), \quad m_1 = 0.$

10. $x\, d^2y/dx^2 + dy/dx + x = 0, \quad (1, -1/4), \quad m_1 = -1/2.$

11. $2\,x + 2\,y\, dy/dx = 3, \quad (1/2, 1), \quad \Delta x = 0.4.$

12. $2\,y\, dy/dx - 3\,x^2 + 1 = 0, \quad (1, 0), \quad \Delta x = 0.5.$

13. $y\, dy + (1 - x^2)dx = 0$, by trajectories.

14. $y^2\, dx - 2\,x\, dy = 0$, by trajectories.

15. $dy/dx = 1/x + 1/y, \quad (1, 1).$

16. $dy/dx = \sqrt{x^2 + y^2}, \quad (2, 1).$

ADDITIONAL PROBLEMS

Solve each of the following equations. (Nos. 1–31.)

1. $d^3y/dx^3 = \sin x.$ *Ans.* $y = \cos x + c_1x^2 + c_2x + c_3.$

2. $d^3y/dx^3 = e^{2x} + x.$

3. $d^3y/dx^3 = x - \cos x.$
Ans. $y = x^4/4\,! + \sin x + c_1x^2 + c_2x + c_3.$

4. $d^2y/dx^2 - y\, dy/dx = 0.$

5. $d^2y/dx^2 - dy/dx + 1 = 0.$ *Ans.* $y = c_1e^x + x + c_2.$

6. $x\, d^2y/dx^2 + dy/dx + x = 0.$

7. $d^2y/dx^2 = (dy/dx)^2 + 1.$ *Ans.* $y = \log \sec (x + c_1) + c_2.$

8. $y\, d^2y/dx^2 - (dy/dx)^2 = y^2.$

9. $(dx/dy)^2 - 1 = x\, d^2x/dy^2.$ *Ans.* $c_1x + \sqrt{c_1{}^2x^2 + 1} = e^{c_1(y+c_2)}.$

10. $d^2y/dx^2 = \sqrt{1 + (dy/dx)^2}.$

11. $(1 + y)d^2y/dx^2 - (dy/dx)^2 = 0.$ *Ans.* $\log (1 + y) = c_1x + c_2.$

12. $2\,y\, d^2y/dx^2 + 2(dy/dx)^2 = y\, dy/dx$ for $dy/dx = 3$ at $(0, 2)$.

13. $x\, d^2y/dx^2 = \sqrt{1 + (dy/dx)^2}$ for $dy/dx = 0$ at $(1, 1/4)$.
Ans. $y = (1/2)(x^2/2 - \log x).$

14. $x\, d^2y/dx^2 + dy/dx = x$ for $dy/dx = 3/2$ at $(1, 0)$.

15. $2x\, dy/dx \cdot d^2y/dx^2 = (dy/dx)^2 - 1$ for $dy/dx = 2$ at $(1, 1)$.
Ans. $9y = 2(3x + 1)^{3/2} - 7$.

16. $d^2s/dt^2 + 4s = \cos 2t$.

17. $d^2s/dt^2 - ds/dt = t$. *Ans.* $s = c_1 + c_2e^t - t - t^2/2$.

18. $d^2x/dy^2 + dx/dy - 2x = ye^y$.

19. $d^2y/dt^2 + y = \sec t$.
Ans. $y = c_1 \cos t + c_2 \sin t + \cos t \log \cos t + t \sin t$.

20. $d^2y/dx^2 + 8\, dy/dx + 15y = 2e^{-3x}$.

21. $d^2s/dt^2 - 3\, ds/dt + 2s = e^{2t} - 3t$.
Ans. $s = c_1e^{2t} + c_2e^t + te^{2t} - 3t/2 - 9/4$.

22. $d^2x/dt^2 - 6\, dx/dt + 9x = e^{3t}/t^2$.

23. $4\, d^2y/dx^2 + y = 0$, if $dy/dx = 3$ at $x = \pi$, $y = 2$.
Ans. $y = 2 \sin (1/2)x - 6 \cos (1/2)x$.

24. $d^2y/dx^2 - 4\, dy/dx + 5y = 0$ for the curve through $(0, 0)$ with slope 1.

25. $d^2y/dx^2 - 6\, dy/dx + 9y = 2e^{3x} + \sin 2x$.
Ans. $y = c_1e^{3x} + c_2xe^{3x} + x^2e^{3x} + (5 \sin 2x + 12 \cos 2x)/169$.

26. $\begin{cases} dx/dt - dy/dt = e^{2t}, \\ d^2y/dt^2 = 2\, dx/dt - x + e^t. \end{cases}$

27. $d^3y/dx^3 + 2\, d^2y/dx^2 + 5\, dy/dx = x$.
Ans. $y = c_1 + e^{-x}(c_2 \sin 2x + c_3 \cos 2x) + x^2/10 - 2x/25$.

28. $d^2y/dx^2 + \tan x\, dy/dx = \sin 2x$.

29. $d^2y/dx^2 - 5\, dy/dx + 6y = e^{nx}$, $(n \neq 2, 3)$.
Ans. $y = c_1e^{2x} + c_2e^{3x} + e^{nx}/(n^2 - 5n + 6)$.

30. $d^4y/dx^4 - 2\, d^3y/dx^3 + d^2y/dx^2 = x$.

31. $(1 - x)d^2y/dx^2 + x\, dy/dx - y = (1 - x)^2$. (Find particular integrals by inspection for the left-hand member set equal to zero and then use variation of parameters.) *Ans.* $y = c_1x + c_2e^x + x^2 + 1$.

32. *Curve of Pursuit.* A point Q starts at the origin and moves along the x axis with constant speed u. At the same time a second point P starts at $A(0, a)$ and moves at a constant speed u/k in a direction always toward Q. Find the equation of the path of P and, assuming $k < 1$, the point at which P catches Q.

CHAPTER XXII

APPLICATIONS OF DIFFERENTIAL EQUATIONS

Some Applications in Statics

237. The Form of Suspended Cables. Suppose the cable of Fig. 231 to be in equilibrium under the action of its weight and tensions T_0 at two points A and B on a horizontal line. Our problem is to determine its form.

Take the y axis vertical and through the lowest point C of the cable. Any segment CP is in equilibrium under the tangential tensions H and T at C and P, respectively, and the weight ws, where w is the weight of the cable per unit length and s is the length of the arc CP. Since CP is in equilibrium, the horizontal components of force are equal, that is,

$$T \cos \theta = H, \tag{1}$$

and the equality of vertical components gives

$$T \sin \theta = ws. \tag{2}$$

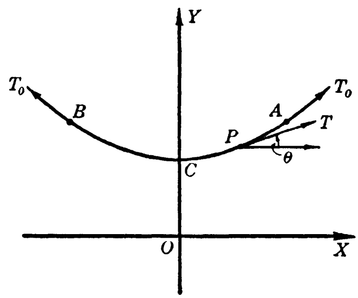


Fig. 231

On dividing (2) by (1) we have

$$\tan \theta = \frac{dy}{dx} = \frac{w}{H} \cdot s, \tag{3}$$

from which, by differentiation,

$$\frac{d^2y}{dx^2} = \frac{w}{H} \cdot \frac{ds}{dx}. \tag{4}$$

But (§ 101)

$$\frac{ds}{dx} = \pm \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

and therefore

$$\frac{d^2y}{dx^2} = + \frac{w}{H} \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

the positive sign being chosen because the curve is concave upward.

Substituting $dy/dx = p$ and integrating, we get

$$\log\,(p + \sqrt{p^2 + 1}) = \frac{w}{H} \cdot x + k,$$

where $k = 0$, since $p = 0$ when $x = 0$. Solving for p, we find

$$p = \frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{w}{H}\cdot x} - e^{-\frac{w}{H}\cdot x}\right),$$

from which

$$y = \frac{H}{2\,w}\left(e^{\frac{w}{H}\cdot x} + e^{-\frac{w}{H}\cdot x}\right) + k_2,$$

or

$$y = \frac{H}{w}\cosh\left(\frac{w}{H} \cdot x\right) + k_2.$$

To fix the position of the x axis, we take $k_2 = 0$; then $y = H/w$ when $x = 0$, or OC is equal to the length of the cable which has as its weight the horizontal tension at the lowest point C. The curve

$$y = \frac{H}{2\,w}\left(e^{\frac{w}{H}\cdot x} + e^{-\frac{w}{H}\cdot x}\right)$$

assumed by the cable is a ***catenary.***

Now consider the situation if the cable has vertical rods attached which suspend a road-bed AB carrying a uniformly distributed load. Let the weight of the road-bed and load be large compared with the weight of the cable and vertical rods. Disregarding the weight of the cable and rods, show that the cable of Fig. 232 takes the form of the parabola $y = wx^2/2\,H$. Here w is the weight of the road-bed and load per foot of road-bed, and H is the horizontal tension at its lowest point.

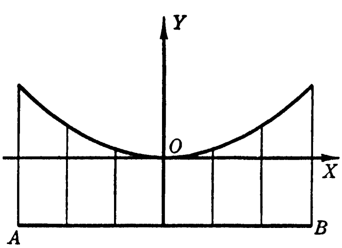


FIG. 232

PROBLEMS

1. If along the roadway of a suspension bridge, $w = 2000$ lbs. per ft. per cable and if $H = 2{,}500{,}000$ lbs., derive the equation of the cable.

Ans. $y = x^2/2500$.

2. If the cable of Problem 1 sags 100 ft. at its mid-point, what is the span length?

3. What is the direction of the cable of Problem 1 at the ends? At the quarter span points? *Ans.* $\pm 2/5$; $\pm 1/5$.

238. Bending of Columns and Beams. When a cantilever beam (one supported at only one end) is bent as in Fig. 233, the fibers on its upper part are stretched and those on the lower are compressed. There is a curve along which the fibers are neither compressed nor extended. This curve is called the ***neutral axis.*** In mechanics it is shown that $M = EI/R$ where M is the bending moment at any point of the axis $P(x_1, y_1)$, I is the moment of inertia of a cross-section of the beam about an axis through P and perpendicular to the xy plane, E is Young's modulus for the material, and R is the radius of curvature of the neutral axis at P.

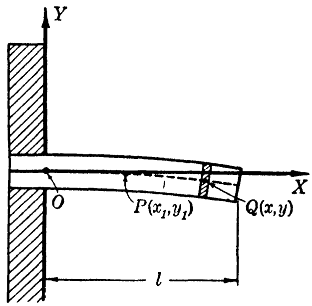


FIG. 233

But $R = [1 + (dy/dx)^2]^{3/2}/(d^2y/dx^2)$ may be replaced approximately by $1/(d^2y/dx^2)$ if dy/dx is small, which gives the approximate relation

$$EI\frac{d^2y}{dx^2} = M. \tag{1}$$

PROBLEMS

1. For a concentrated load W lbs. at the end of the cantilever beam shown above, the bending moment at P is $W(l - x_1)$. Derive the equation of the neutral axis (elastic curve).

2. Determine the deflection at the free end of the neutral axis in Problem 1.

3. Find the slope of the elastic curve of Problem 1 at its free end-point.

4. A timber cantilever beam 4 in. by 4 in. in cross-section projects 120 in. from the face of a brick wall. A load of 200 lbs. is placed at the end. If $E = 1{,}500{,}000$ lbs. per sq. in. and $I = 64/3$, determine the equation of the

elastic curve. What is the deflection at the free end? Also what is the slope of the curve at the free end?

5. Let a beam of homogeneous material and constant cross-section be fastened horizontally at one end. Also suppose a load distributed over it uniformly such that the weight of load and beam is w lbs. per foot. Then the bending moment at P due to the shaded element at Q is $w(x - x_1)dx$, and the total bending moment at P due to the part from P to the end of the beam is (Fig. 233)

$$M = w\int_{x_1}^{l}(x - x_1)dx = \frac{w}{2}(l - x_1)^2.$$

Hence, from (1),

$$\frac{d^2y_1}{dx_1^2} = -\frac{w}{2\,EI}(l - x_1)^2.$$

The negative sign is used because the curve is concave downward. Find y_1 and show that the deflection is $wl^4/8\,EI$ at the end.

6. Let the weight of a beam be w lbs. per foot and let a weight of Q lbs. be applied at the end. Show that

$$EI\frac{d^2y}{dx^2} = -\frac{w}{2}(l - x)^2 - Q(l - x).$$

Find y and the deflection at the free end.

7. The beam OAC of homogeneous material and uniform cross-section has forces W applied at the center of its ends. Find an expression for the force which will produce bending.

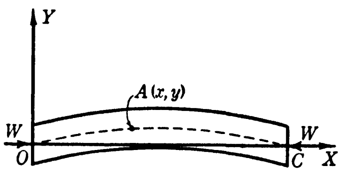


Fig. 234

Disregarding the weight, we have

$$EI\frac{d^2y}{dx^2} = -Wy,$$

from which

$$y = c\sin\left(\sqrt{\frac{W}{EI}}\,x\right),$$

since $y = 0$ when $x = 0$. Evidently c represents the maximum deflection, and is not zero if bending occurs. Then, since $dy/dx = 0$ at $x = l/2$,

$$0 = c\sqrt{\frac{W}{EI}}\cos\left(\sqrt{\frac{W}{EI}}\cdot\frac{l}{2}\right),$$

which means that

$$\sqrt{\frac{W}{EI}}\cdot\frac{l}{2} = \frac{\pi}{2},$$

or

$$W = \pi^2\frac{EI}{l^2}$$

if bending occurs. This last formula is known as ***Euler's formula*** for the strength of a column and may also be obtained as follows. The value of x which locates C is l and it locates the end of an arch of the sine curve. Hence $\sqrt{W/EI} \cdot l = \pi$ or $W = \pi^2 EI/l^2$.

8. If a beam is fastened at both ends, there must act at each end a bending moment k to keep the slope at those points zero. Then, in addition to the moment Wy which tends to produce bending, there exists a moment k opposing bending. Therefore

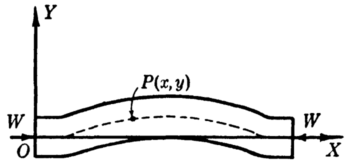


FIG. 235

$$EI\frac{d^2y}{dx^2} = k - Wy.$$

Show that $y = (k/W)[1 - \cos(\sqrt{W/EI}x)]$ and that $W = 4\pi^2 EI/l^2$ will produce bending.

SOME APPLICATIONS TO DYNAMICS

239. Simple Pendulum. An important case of motion in a plane is that of a simple pendulum. Assume the bob to be suspended by a very light wire or rod, the weight of which we shall neglect. Resolving forces along the tangent at B and disregarding the resistance of the air, we get

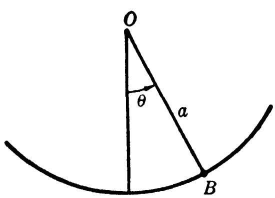


FIG. 236

$$m\frac{d^2s}{dt^2} = -mg\sin\theta,$$

where $s = a\theta$.

If θ is kept so small that $\sin\theta$ is approximately equal to θ, this gives

$$\frac{d^2\theta}{dt^2} = -\frac{g\theta}{a},$$

the solution of which is

$$\theta = c_1\sin\left(\sqrt{\frac{g}{a}}\cdot t\right) + c_2\cos\left(\sqrt{\frac{g}{a}}\cdot t\right).$$

If $\theta = 0$ when $t = 0$ and if θ_0 is the maximum displacement, then

$$\theta = \theta_0\sin\left(\sqrt{\frac{g}{a}}\cdot t\right).$$

At the maximum displacement

$$\sqrt{\frac{g}{a}} \cdot t = \frac{\pi}{2}, \qquad \text{or} \qquad t = \frac{\pi}{2}\sqrt{\frac{a}{g}},$$

from which the period is $T = 2\pi\sqrt{a/g}$.

For the ***seconds pendulum***, this makes $a = g/4\pi^2$.

240. Non-Periodic Motion of Bodies in Resisting Media. When a body moves in a medium, that medium offers a resistance depending on the velocity, but the exact law of dependence is not known.

EXAMPLE

Suppose a body falling from rest in air meets with resistance proportional to the velocity. Let the resistance equal c when the velocity is unity. Then, since effective accelerating force equals the impressed force minus the resistance, we have

$$m\frac{dv}{dt} = mg - cv,$$

or

$$\frac{dv}{dt} = g - kv,$$

where $k = c/m$. This gives

$$\frac{ds}{dt} = v = \frac{g(1 - e^{-kt})}{k},$$

since $v = 0$ when $t = 0$. Then

$$s = \frac{g}{k^2}(kt + e^{-kt} - 1),$$

since $s = 0$ when $t = 0$.

Observe that as t increases $v \to g/k$ and $s \to (gt/k - g/k^2)$, which are expressions for motion with a constant velocity.

PROBLEMS

1. Suppose that in the example above the resistance of the air had varied as the square of the velocity. Then $dv/dt = g - kv^2$. Show that

$$v = \sqrt{\frac{g}{k}} \cdot \frac{e^{t\sqrt{gk}} - e^{-t\sqrt{gk}}}{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}} \qquad \text{and} \qquad s = \frac{1}{k}\log\left(\frac{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}}{2}\right).$$

Observe that as t increases $v \to \sqrt{g/k}$ and $s \to (1/k)(t\sqrt{gk} - \log 2)$, which are expressions for motion with a constant velocity.

2. A constant force E begins to act at time $t = 0$ on a boat of mass M. Assuming that the water offers a resistance which is proportional to the velocity v and which equals R when the velocity is unity, find v in terms of t. The effective accelerating force is Mdv/dt, the impressed force is E, and the resisting force is Rv. Therefore

$$M\frac{dv}{dt} = E - Rv.$$

Note that $v \to E/R$ as t increases and that this is a particular solution of the differential equation.

3. Suppose in the problem above that the boat is moving with a velocity v_0 when $t = 0$ and the propelling force E is suddenly removed. Find expressions for the velocity and the distance the boat will move.

241. Periodic Motion of Bodies in Media. A mass M is supported by a spiral spring fixed at the end A. Let O be the position of equilibrium of the mass and let the stiffness of the spring be such that a force of 1 lb. elongates it c ft. Disregarding the resistance of the air, the force on M is equal to $-q/c$, by Hooke's law, where q is the displacement from the position of equilibrium. Therefore

$$(1)\qquad M\frac{d^2q}{dt^2} = -\frac{q}{c},$$

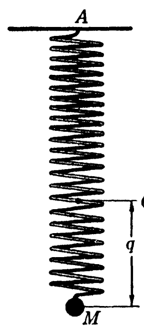


FIG. 237

which has the solution

$$q = k_1 \sin\frac{t}{\sqrt{Mc}} + k_2 \cos\frac{t}{\sqrt{Mc}}\cdot$$

If the motion arose from releasing the mass at time $t = 0$ after it had been pulled a ft. below its position of equilibrium, then $k_1 = 0$, $k_2 = a$, and

$$q = a\cos\frac{t}{\sqrt{Mc}}\cdot$$

The motion is simple harmonic of period $2\pi\sqrt{Mc}$. We see from the expression for the period that the ***frequency*** of the vibrations is increased by decreasing the mass or increasing the stiffness of the spring.

Let the mass M be immersed in a fluid which offers a resistance

proportional to the velocity and which equals R when the velocity is unity. Then the resisting force is

$$-\frac{q}{c} - R\frac{dq}{dt}$$

and the equation becomes

$$M\frac{d^2q}{dt^2} = -\frac{q}{c} - R\frac{dq}{dt}, \tag{2}$$

a *homogeneous linear equation* with constant coefficients.

In the two cases just considered the system when once started in motion was left to itself; in such cases, vibrations are called ***free oscillations.***

Let us now suppose that, in addition to the frictional force and the stiffness of the spring, an outside force E acts on M. Then the equation becomes

$$M\frac{d^2q}{dt^2} = E - \frac{q}{c} - R\frac{dq}{dt}, \tag{3}$$

a *complete linear equation* with constant coefficients. In this case, the vibrations are called ***forced oscillations***, provided E is a periodic function of the time.

242. Free Oscillations. If $R^2c > 4\,M$, the solution of equation (2) in § 241 is

$$q = e^{-Rt/(2M)}(m_1e^{kt} + m_2e^{-kt}), \tag{a}$$

where $k = \sqrt{(R^2c - 4\,M)/4\,M^2c}$. If $R^2c = 4\,M$, the solution is

$$q = e^{-Rt/(2M)}(at + b). \tag{b}$$

If $R^2c < 4\,M$, the solution is

$$q = e^{-Rt/(2M)}[m_1 \sin (lt) + m_2 \cos (lt)], \tag{c}$$

where $l = \sqrt{(4\,M - R^2c)/4\,M^2c}$.

In case (c), the motion is that of free oscillations of period $4\,\pi M\sqrt{c}/\sqrt{4\,M - R^2c}$, which have a decreasing amplitude, or are damped, due to the factor $e^{-Rt/(2M)}$. The period may be increased by making $R^2c \rightarrow 4\,M$.

In cases (a) and (b), the motion is non-oscillatory.